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**A GEOMETRIC METHOD FOR THE PERIODIC
PROBLEM IN ORDINARY DIFFERENTIAL
EQUATIONS**

par

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DIFFERENTIAL EQUATIONS

by

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MOT DES ÉDITEURS

Cette monographie a été rédigée par le professeur Srzednicki lors de son séjour au DMI de l'Université de Sherbrooke du 1^{er} février au 15 mai 1992. Elle met à la portée du lecteur non initié les résultats de sa thèse de doctorat d'état (en Pologne). Une partie du contenu de ce travail a été exposée au département dans le cadre du séminaire d'analyse.

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PREFACE

In this monograph we present foundations and basic applications of a new method for the periodic problem for ordinary differential equations. The method was introduced in [Sr2,3] and is based on the Lefschetz Fixed Point Theorem and the Ważewski Topological Principle.

The paper consists of two parts, each of which is divided into chapters. The chapters are in turn divided into sections. In Part I we introduce necessary definitions and prove the main theorems concerning the geometric method, while in Part II we apply the method to various classes of ordinary differential equations.

The geometric method is based on the concept of periodic block (see Definition 2.2.4), which is a modification of the notion of isolating block, introduced by C.Conley. Its intuitive description is presented in Section 1.1, but its rigorous treatment is postponed until Section 2.2. Some motivation for that concept comes also from the Floquet theory, as we shall see in Section 3.3. In Section 3.2 we present a construction of periodic blocks for ordinary differential equations. The construction is further developed in Section 3.4, where Proposition 3.4.1 is stated. That result enables us to determine which equations admit periodic blocks and is used in many applications.

A most important role in the method is played by the Lefschetz number, an invariant associated to periodic blocks. It

is described in Sections 1.1 and 2.2 (see Definition 2.2.3).

Theorems A, B, and C established in Section 1.2 are the main results for the method. They are formulated in a purely topological way, without referring to any differential structure. Section 2.3 contains their proof. In Section 4.2 we modify them, and the resulting Theorems 4.2.1 and 4.2.2 apply directly to ordinary differential equations. They might be compared with other results on the periodic problem, recalled in Section 4.1.

At first we apply the geometric method to perturbed linear equations in Section 5.1. Proposition 5.1.3 is the most original result in that section. Its proof, as well as the proofs of Propositions 5.1.1. and 5.1.2, represents a standard way for applying the method. In Section 5.2 we establish a relation between the Conley index and the fixed point index. The most important applications of the method presented in this paper are contained in Chapter 6. Theorems 6.3.1 and 6.3.3 on planar polynomial equations and Theorem 6.7.1 on rational equations are the main results of Chapter 6, which is based on the papers [Sr4] and [KS]. The theorems concern nonautonomous equations obtained from hyperbolic and elliptic polynomials (compare Definition 6.2.2), and are motivated by some geometric observations given in Section 6.1. Several examples illustrating them are presented in Sections 6.3 and 6.7. In Section 7.1 we apply the geometric method to results on second order scalar equations. Section 7.2 is based on [SS] and presents two results on scalar equations of arbitrary order.

Unless otherwise is stated, all definitions and theorems concerning the geometric method as well as results obtained by their applications presented in this paper were established in [KS], [Sr1-4], and [SS] or appear here for the first time.

PART I. FOUNDATIONS OF THE GEOMETRIC METHOD.

Chapter 1. Introduction to the method.

We begin with an informal introduction to the geometric method. In Section 1.1 we give intuitive ideas of the basic notions of this paper - rigorous presentation is left for Chapter 2. Theorems A, B, and C, the main results for the method, are stated in Section 1.2. Section 1.3 contains four simple examples of their applications. Concluding remarks concerning further investigations of the method are presented in Section 1.4.

1.1. Basic notions. This paper is devoted to presentation of a new topological method for the boundary value problem

$$(1.1) \quad \dot{x} = f(t, x),$$

$$(1.2) \quad x(0) = x(T),$$

where f is continuous and T -periodic in t . The problem (1.1), (1.2) is called the periodic problem. Its solutions coincide with T -periodic solutions of (1.1).

Usual methods for the problem (in the case of nonlinear f) apply degree theory both in the finite dimensional setting (like the Krasnosel'skii method of guiding functions, compare [K], [KZ], [RM]) and in the functional analytic framework (see [KZ], [Ma], [RM]). We recall some theorems concerning those methods in Section 4.1. By application of the fixed point index, results of [CMZ] and [CZ1,2] extend those theorems to the case in which the phase space is not an open subset of a vector space. For extensive

bibliographies concerning the problem we refer the reader to [CMZ], [CZ2], [KZ], [Ma], and [RM].

The method presented here was established in [Sr3] and is based on improvement of ideas introduced in the paper [Sr2]. The main theorem on the method enables to compute the fixed point index of the Poincaré operator associated to solutions contained in periodic blocks. A proof of the theorem applies the Lefschetz Fixed Point Theorem and a version of the Theorem of Ważewski from the theory of isolated invariant sets (see [C2] or [R]).

As in [H], we apply the notion of local process (see Definition 2.1.1) to present essential properties of solutions of a non-autonomous equation under the assumption of uniqueness of the associated Cauchy problem. The local process φ generated by the equation (1.1) is a collection $\{\varphi_{(\sigma,t)}\}_{\sigma,t}$ of maps of the phase space into itself such that the function $\tau \longrightarrow \varphi_{(t_0, \tau-t_0)}(x_0)$ is the solution of (1.1) with the initial condition x_0 at time t_0 . In other words, the map $\varphi_{(\sigma,t)}$ describes the evolution of the phase space along the solutions of the equation from time σ to time $\sigma+t$. If the equation is autonomous (i.e. independent of t) then $\varphi_{(\sigma,t)}$ is independent of σ , hence it is the time- t map of the local flow generated by the equation. Thanks to that concordance, processes generalize flows to the non-autonomous case in the most direct way. If the right-hand side of the equation is T -periodic in the time-parameter then the generated local process is T -periodic (still compare Definition 2.1.1).

Let φ be a local process on a topological space X . By a trajectory of φ through $(\sigma, x) \in \mathbb{R} \times X$ we mean the graph of the function $\tau \longrightarrow \varphi_{(\sigma, \tau - \sigma)}(x)$ (i.e the graph of the solution through (σ, x) , provided φ is generated by a differential equation). A trajectory is called T -periodic if the corresponding function is T -periodic.

For each subset Z of $\mathbb{R} \times X$, by Z^- we denote its exit set (with respect to φ). It consists of points (t, x) in Z , such that for arbitrary small positive ε , the segment of the trajectory through (t, x) over the interval $[t, t + \varepsilon)$ is not contained in Z (the definition, due to C. Conley, is presented in (2.3)). By $Z(t)$ we denote the t -section of Z , i.e the set $\{x \in X: (t, x) \in Z\}$. In this paper we do not consider arbitrary subsets of $\mathbb{R} \times X$, our interest is focused on sets which we call T -periodic proper. $Z \subseteq \mathbb{R} \times X$ is such a set if the projection $Z \longrightarrow \mathbb{R}$ is a trivial bundle (hence Z is its total space, \mathbb{R} is the base and $Z(0)$ is the fiber), $Z(0)$ is compact (thus $Z(t)$ is also compact for each t) and $Z(t+T) = Z(t)$. Figure 1 depicts an example of a T -periodic proper set for $X = \mathbb{R}^2$. Here Z is not connected, its components look like the spirals of a DNA-molecule.

Actually, we do not deal with single sets but rather with pairs of sets (in fact, with the pairs of the form (Z, Z^-)). Roughly speaking, a T -periodic proper pair is the total space of a pair of trivial bundles, each of the bundles satisfies the above conditions. A single set Z is always treated as the pair (Z, \emptyset) . In all examples considered in this paper, the bundle structure for a

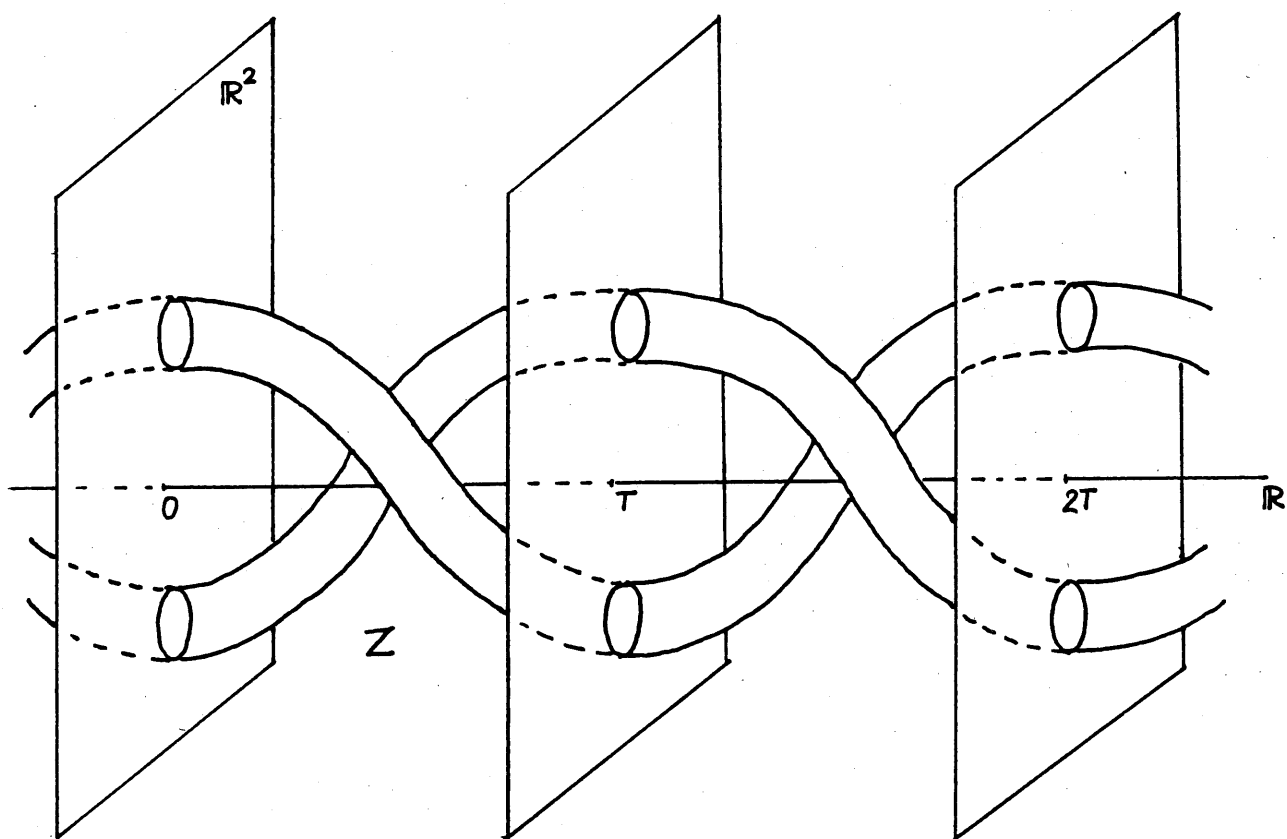


Figure 1

T-periodic proper pair is generated by a process on the space X , i.e. both the elements of the pair consist of trajectories of the process (the process does not need to be T-periodic). Therefore, in order to make the presentation more comprehensive, we decided to introduce the definition of the T-periodic pair in terms of processes only (compare Definition 2.2.2), without referring to the approach based on bundle structures (that approach is formally more general).

Let (A,B) be a T-periodic proper pair. By Definition 2.2.2, both A and B can be filled by the trajectories of some process ω . By our assumptions, $\omega_{(t,T)}$ maps the pair $(A(t),B(t))$ into itself, its restriction to the pair is called a monodromy map. If we additionally assume, that both $A(t)$ and $B(t)$ are euclidean neighborhood retracts (ENRs; one can assume - equivalently - that A and B are ENRs), the Lefschetz number of the monodromy map is defined. It appears that that number is an invariant which does not depend on the choice of ω and t , it is called the Lefschetz number of (A,B) and is denoted by $\text{Lef}_T(A,B)$ (compare Definition 2.2.3). The role of T in that notation is important, (A,B) is a kT -periodic pair as well but its Lefschetz numbers for various k can differ. As an example of such a situation, consider the set Z from Figure 1. Here $\text{Lef}_T(Z) = 0$ but $\text{Lef}_{2T}(Z) = 2$.

Let us again consider a local process φ on X . A set $W \subseteq \mathbb{R} \times X$ is called a T-periodic block for φ (compare Definition 2.2.4) if (W, W^-) is a T-periodic proper pair of ENRs. Moreover, if there are no trajectories of φ , which are contained in W and intersect ∂W ,

then W is called an isolating T -periodic block.

1.2. The main theorems. In the sequel we will assume that φ is a local T -periodic process on X . The geometric method is based on the following abstract Theorems A, B, and C.

Theorem A. Let W be a T -periodic block for φ . If

$$\text{Lef}_T(W, W^-) \neq 0$$

then there is a T -periodic trajectory of φ contained in W .

In the next two theorems we assume additionally that X is an ENR. Theorem B plays the central role in this paper. Actually, Theorem A is a consequence of Theorem B, provided the considered periodic block is isolating.

Theorem B. Let W be an isolating T -periodic block for φ . Then the set

$$K_W = \{x \in X: \varphi_{(0,T)}(x) = x, \forall t \in [0, T]: \varphi_{(0,t)}(x) \in W(t)\}$$

is compact and isolated in the set of fixed points of $\varphi_{(0,T)}$ (hence the fixed point index, denoted here by $\text{ind}(\varphi_{(0,T)}, K_W)$, is determined) and

$$\text{ind}(\varphi_{(0,T)}, K_W) = \text{Lef}_T(W, W^-).$$

Recall that the fixed point index is defined in [D, VII.5].

The map $\varphi_{(0,T)}$ considered above is called the Poincaré (or Poincaré-Andronov) operator of φ . In an obvious way its fixed

points are in a one-to-one correspondence with T -periodic trajectories of φ . The set K_W is contained in some isolated invariant set for the discrete flow $\{\varphi_{(0,T)}^n\}_{n \in \mathbb{Z}}$, hence results of the M.Mrozek's paper [M3] also apply to it. However, our pair $(W(0), W^-(0))$ is not an index pair in the Mrozek's sense, hence Theorem B does not follow from [M3, Th.4]. We mention, that [CMZ, Cor.2], [KZ, Th.28.5] and [Ma, Cor.III.12] represent other results on invariants of that operator, which are equivalent to the fixed point index. Proofs of Theorems A and B are contained in Section 2.3.

Many applications of the method presented in this paper are based on the following immediate consequence of Theorem B:

Theorem C. Assume that W_1 and W_2 are T -periodic isolating blocks for φ , $W_1 \subseteq W_2$, and let

$$\text{Lef}_T(W_1, W_1^-) \neq \text{Lef}_T(W_2, W_2^-).$$

Then there is a T -periodic trajectory of φ which is contained in $\text{int } W_2$ and is not contained in W_1 . ■

1.3. Some examples. Let us present some simple examples which indicate the way in which Theorems A, B, and C can be applied.

At first consider the equation

$$(1.3) \quad \dot{z} = \bar{z}^2 e^{it} + 1$$

on the complex plane. This equation, as well as similar ones, is analyzed in Example 6.4.1 (see also Section 6.1). It can be proved that the local process generated by (1.3) has a 2π -periodic

isolating block W which has the form of a twisted prism having hexagonal base (see Figure 2). Its exit set W^- consists of three disjoint ribbons winding around the prism. By the picture it is not difficult to conclude that

$$\text{Lef}_{2\pi}(W, W^-) = \text{Lef}_{4\pi}(W, W^-) = 1, \text{Lef}_{6\pi}(W, W^-) = -2.$$

Theorem A immediately imply, that (1.3) has a 2π -periodic solution. That result does not follow directly from the standard methods on the periodic problem. The method of guiding functions (see Theorem 4.1.2) cannot be applied, simply because the equation does not have any guiding function at all. Indeed, for each sufficiently large z the right-hand side of (1.3) at z makes the full rotation as t varies from 0 to 2π , hence its scalar product with the gradient of any function cannot be always positive. Actually, it follows also that we cannot associate any strict bound set to our problem (see remarks in Section 4.1). Theorem of Mawhin in its simplest formulation (with constant homotopy, see Theorem 4.1.3) also is not applicable, at least because the averaged vector-field of (1.3) is constant. However, Theorem 4.1.4 due to A.Capietto, J.Mawhin, and F.Zanolin applies also to (1.3), it can be proved that there is a continuation, leaving the set of 2π -periodic solutions bounded, to the linear equation $\dot{z} = \frac{1}{3}iz$. (It seems not to be so obvious, some constructions introduced below lead to the proof.) For that reason let us consider a modification of (1.3):

$$(1.4) \quad \dot{z} = \frac{2}{z}e^{it} + e^{it/3}.$$

The local process of (1.4) is 6π -periodic and admits the isolating

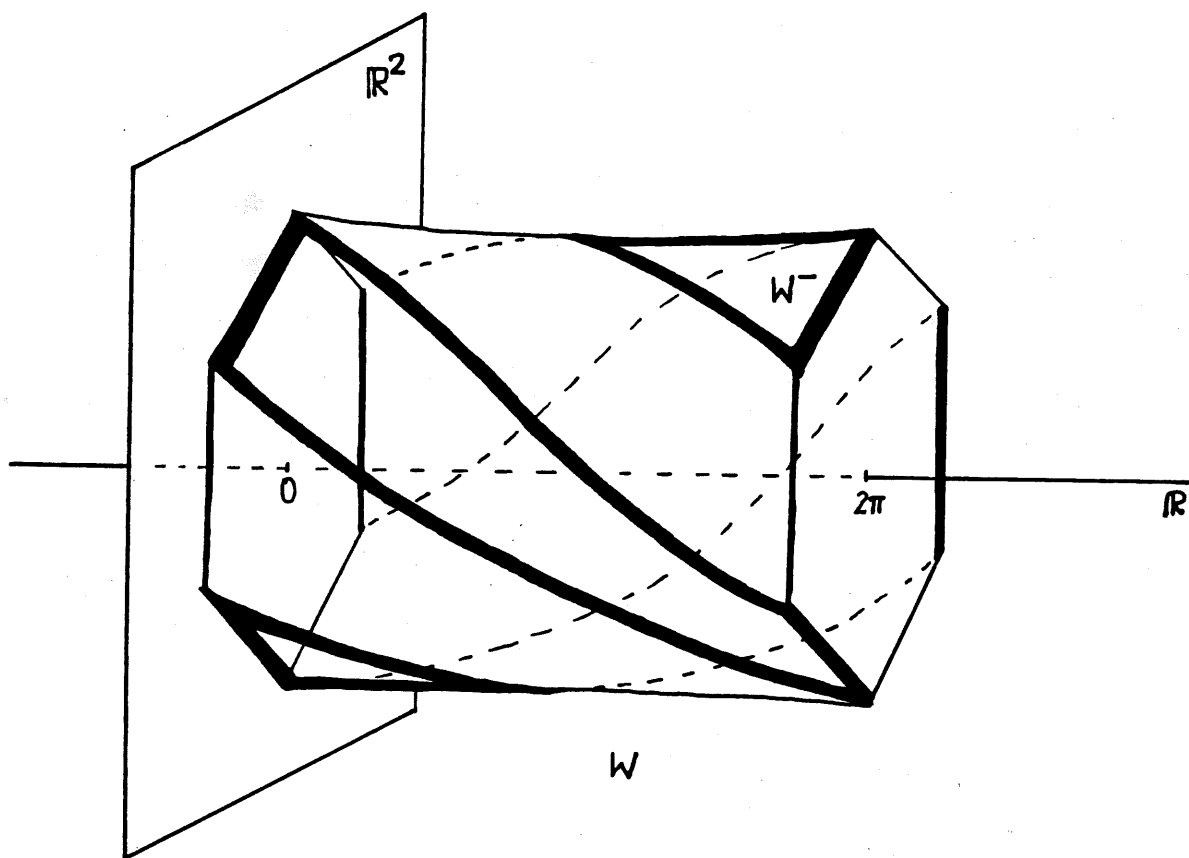


Figure 2

periodic block W from Figure 2 (it should be large enough). As before, by the calculation of its Lefschetz number, Theorem A implies the existence of a 6π -periodic solution of (1.4). Exactly the same remarks on the guiding functions, strict bound sets and the Mawhin's theorem, which above were stated for (1.3), are valid for (1.4). However, in that example there is no longer required continuation to any linear system. Indeed, for such a system the fixed point index of the Poincaré map at the origin has to be equal to ± 1 , but in the case of (1.4) the related index is equal to -2 by Theorem B.

Let us consider another modification of (1.3):

$$(1.5) \quad \dot{z} = \bar{z}^2 e^{it} + z.$$

Zero is a periodic solution of it, so the problem is to find another one. Equations of that form are presented in Example 6.4.4. The local process generated by (1.5) admits a 2π -periodic block U which has the form of the (sufficiently large) set W on Figure 2 having removed the cartesian product of the real line and a small disc centered in the origin. The exit set of U is the same as previously, i.e. consists of the three ribbons. One can calculate, that

$$\text{Lef}_{2\pi}(U, U^-) = \text{Lef}_{4\pi}(U, U^-) = 0, \text{Lef}_{6\pi}(U, U^-) = -3.$$

We apply Theorem A again. As a result we obtain the existence of a 6π -periodic solution of (1.5). Actually there are at least three distinct such solutions, because (1.5) is symmetric with respect to the rotation of the angle $2\pi/3$ around the origin.

We modify once more the initial equation:

$$(1.6) \quad \dot{z} = \bar{z}^2 e^{it} + \bar{z}$$

(compare Example 6.4.6). There are two isolating 2π -periodic blocks for the local process of (1.6). The larger one, denoted W_2 , is essentially the same as W above (again it should be chosen sufficiently large). It follows by the previous calculations that $\text{Lef}_{2\pi}(W_2, W_2^-) = 1$. The smaller one, W_1 , is equal to the cartesian product of \mathbb{R} and the square $\{z: |\text{Re } z| \leq \delta, |\text{Im } z| \leq \delta\}$, for some $\delta > 0$ small enough. Its exit set W_1^- is equal to $\mathbb{R} \times \{z: |\text{Re } z| = \delta, |\text{Im } z| \leq \delta\}$, hence $\text{Lef}_{2\pi}(W_1, W_1^-) = -1$. By Theorem C we obtain at once that the considered equation has a nontrivial 2π -periodic solution. Again there is no hint how to achieve those results on (1.5) and (1.6) by usual methods directly. Theorem C is in the spirit of [KZ, Th.13.5], but the latter one is useless for (1.6) by lack of guiding functions, as in the case of (1.3) and (1.4) analyzed above.

The right-hand sides of (1.3) - (1.6) are examples of functions which we call polynomials with periodic coefficients in t -variable. It is the simplest class of functions which form the right-hand sides of non-autonomous time-periodic equations. The sum of some first terms of the Fourier-Taylor expansion of each of the functions is in that class. There are many results concerning periodic solutions which can be applied to these polynomials, however they mainly treat the case, in which the highest order monomials (with respect to the space variable) have coefficients independent of time (compare for example [Cr1,2,3], [G] or [L1,2]). As we have shown above, our method enables us to drop

that assumption. Following [Sr4], in Chapter 6 we give a more detailed treatment on the subject.

Chapter 2. Topological background of the method.

In this chapter we introduce abstract results concerning processes in topological spaces. We do not need any differential structure on the considered spaces. In the forthcoming chapters we present how those results can be reformulated in order to obtain theorems concerning differential equations. In Section 2.1 we present the definitions of local processes and related notions, and introduce the concept of composition of local processes. In Section 2.2 we introduce the notions of T -periodic proper pair, its Lefschetz number and T -periodic (isolating) block. Section 2.3 contains proofs of Theorems A and B established in Section 1.2. In Section 2.4 we present two results relating to Theorems A and B.

2.1. Topological processes. Assume that X is a topological space and $\varphi: D \longrightarrow X$ is a continuous mapping, where $D \subseteq \mathbb{R} \times X \times \mathbb{R}$ is an open set. In the sequel we will denote by $\varphi_{(\sigma, t)}$ the function $\varphi(\sigma, \cdot, t)$. The domain of that function is always open (but possibly empty).

Definition 2.1.1. φ is called a local process (on the space X) if the following conditions are satisfied (compare [H, Def. 4.1.1]).

- (i) $\forall \sigma \in \mathbb{R}, x \in X: \{t \in \mathbb{R}: (\sigma, x, t) \in D\}$ is an interval.

$$(ii) \forall \sigma \in \mathbb{R}: \varphi_{(\sigma,0)} = \text{id}_X.$$

$$(iii) \forall \sigma, s, t \in \mathbb{R}: \varphi_{(\sigma, s+t)} = \varphi_{(\sigma+s, t)} \circ \varphi_{(\sigma, s)}.$$

In the case $D = \mathbb{R} \times X \times \mathbb{R}$, we call φ a (global) process. For

$(\sigma, x) \in \mathbb{R} \times X$ the set $\{(\sigma+t, \varphi_{(\sigma, t)}(x)) \in \mathbb{R} \times X: (\sigma, x, t) \in D\}$ is called the trajectory of (σ, x) in φ . If T is a positive number and φ fulfills additionally

$$(iv) \forall \sigma, t \in \mathbb{R}: \varphi_{(\sigma+T, t)} = \varphi_{(\sigma, t)},$$

we call φ a T -periodic local (or global) process.

It follows, that the interval in (i) is open and, by (ii), it contains 0. Since the domains of both the maps in (iii) are equal, $(\sigma, x, s+t) \in D$ if and only if $(\sigma, x, s) \in D$ and $(\sigma+s, \varphi_{(\sigma, s)}(x), t) \in D$. If φ is a process then $\varphi_{(\sigma, t)}: X \longrightarrow X$ is a homeomorphism and $\varphi_{(\sigma+t, -t)}$ is its inverse. If a local (or global) process is independent of the first variable (i.e. is T -periodic for each T), it becomes a local (global, respectively) flow.

A local process φ on X determines a local flow φ^* on $\mathbb{R} \times X$ by the formula

$$(2.1) \quad \varphi_t^*(\sigma, x) = (\sigma+t, \varphi_{(\sigma, t)}(x)).$$

Let us introduce the composition of processes. Assume that ψ and ω are two (local) processes on X . Define

$$(2.2) \quad (\omega \circ \psi)_{(\sigma, t)} = \omega_{(0, t+\sigma)} \circ \psi_{(\sigma, t)} \circ \omega_{(\sigma, -\sigma)}.$$

An elementary verification shows that $\omega \circ \psi$ is also a (local) process. Observe, that even if ψ and ω are flows, their composition is no longer a flow, unless they are commutative. In the sequel we will encounter such situations, they explain once

again our interest in processes.

Assume that φ is a T -periodic local process on X . A point $(\sigma, x) \in \mathbb{R} \times X$ is T -periodic (with respect to φ) if $\varphi_{(\sigma, t+T)}(x) = \varphi_{(\sigma, t)}(x)$ for every $t \in \mathbb{R}$. In order to determine all periodic points, it suffices to look for periodic points in $\{0\} \times X$. A point $(0, x)$ is T -periodic if and only if x is a fixed point of $\varphi_{(0, T)}$ (called the Poincaré map for φ).

2.2. Periodic pairs and blocks. Let π be a local flow on a topological space X . For every subset Z of X we put

$$(2.3) \quad Z^- = \{x \in Z: \exists \varepsilon_n > 0, \varepsilon_n \longrightarrow 0, \pi_{\varepsilon_n}(x) \notin Z\}.$$

Z^- is called the exit set of Z with respect to π .

Let $B \subseteq X$.

Definition 2.2.1. We say that B is a block for π if both the sets B, B^- are compact. The block B is called isolating if

$$\forall x \in \partial B \exists t \in \mathbb{R}: \pi_t(x) \notin B.$$

Our definition of an isolating block is more general than that in [Ch], [Sm, p.463] or [Sr1] and is motivated by the notion of Ważewski set introduced by Conley in [C1] (see also [C2, p.24], [Sr2]). Isolating blocks have appeared in the theory of isolated invariant sets. A set $S \subseteq X$ is an isolated invariant set for π if it is compact and there is a neighborhood N of S such that S is the maximal invariant subset of N . The fundamental result in the theory states, that if X is locally compact and metric, and S is

isolated invariant then in every neighborhood U of S there is an isolating block B such that $S \subseteq B \subseteq U$ (compare [Ch] or [Sm, p.468]). Further remarks on that theory are postponed to Section 5.2.

Our aim is to generalize Definition 2.2.1 to the case of local processes. To this end we introduce the following notation. For every set $Z \subseteq \mathbb{R} \times X$ and $t \in \mathbb{R}$ put

$$Z(t) = \{x \in X: (t, x) \in Z\}.$$

We call the set Z T -periodic ($T > 0$) if for every $t \in \mathbb{R}$

$$Z(t+T) = Z(t).$$

Assume that (A, B) is a pair of subsets of $\mathbb{R} \times X$.

Definition 2.2.2. (A, B) is called a T -periodic proper pair if

- (i) A and B are T -periodic.
- (ii) $A(t)$ and $B(t)$ are compact for each $t \in \mathbb{R}$.
- (iii) There is a process ω on X such that A and B are invariant with respect to the flow ω^* defined in (2.1).

The above condition (iii) means that both A and B consist of trajectories of ω . If (A, B) is a T -periodic proper pair and processes ω and ω' fulfill (iii) then for every $\sigma, t \in \mathbb{R}$

$$\omega_{(\sigma, t)} \simeq \omega'_{(\sigma, t)}: (A(\sigma), B(\sigma)) \longrightarrow (A(\sigma+t), B(\sigma+t)),$$

a homotopy is given by $\omega'_{(\sigma+(1-s)t, st)} \circ \omega_{(\sigma+t, -st)} \circ \omega_{(\sigma, t)}$ for $s \in [0, 1]$. Thus, for each $\sigma \in \mathbb{R}$ the homeomorphism $\omega_{(\sigma, T)}$ of $(A(\sigma), B(\sigma))$ into itself is defined up to homotopy and is called a T -monodromy homeomorphism of (A, B) at σ . T -monodromy

homeomorphisms at σ and $\sigma' \in \mathbb{R}$ are naturally conjugated. It follows in particular, that if (A,B) is a pair of ENR's (euclidean neighborhood retracts) then the Lefschetz number $\text{Lef}(\omega_{(\sigma,T)})$ is defined and does not depend on the choice of σ and ω . (Recall that

$$\text{Lef}(\omega_{(\sigma,T)}) = \sum_{n=0}^{\infty} (-1)^n \text{tr } \omega_{(\sigma,T)*n} \in \mathbb{Z},$$

where $\omega_{(\sigma,T)*n}$ denotes the automorphism of $H_n(A(\sigma), B(\sigma); \mathbb{Q})$ induced by $\omega_{(\sigma,T)}$.) Thus we can introduce the following definition.

Definition 2.2.3. The Lefschetz number of a T-periodic proper pair (A,B) of ENRs (denoted by $\text{Lef}_T(A,B)$) is defined as $\text{Lef}(\omega_{(\sigma,T)})$ for some process ω satisfying (iii) in the Definition 2.2.2 and some $\sigma \in \mathbb{R}$.

By elementary homological algebra it follows that

$$(2.4) \quad \text{Lef}_T(A,B) = \text{Lef}_T(A) - \text{Lef}_T(B)$$

In the simplest case $(A,B) = \mathbb{R} \times (C,D)$, for some compact ENRs C and D contained in X , we have

$$(2.5) \quad \text{Lef}_T(\mathbb{R} \times (C,D)) = \chi(C,D) = \chi(C) - \chi(D),$$

where χ denotes the Euler characteristic.

Assume now, that φ is a local process on X and $W \subseteq \mathbb{R} \times X$. In the sequel by W^- we will denote its exit set with respect to φ^* (see (2.1) and (2.3)). The basic notion of this paper is defined as follows.

Definition 2.2.4. The set W is called a T-periodic block for

φ if (W, W^-) is a T -periodic proper pair of ENRs. If, moreover,

$$\forall (\sigma, x) \in \partial W \exists t \in \mathbb{R}: \varphi_{(\sigma, t)}(x) \notin W(\sigma+t)$$

then W is called an isolating T -periodic block.

Obviously, our additional assumption on the ENR-structure of W and W^- is motivated by the possibility of introduction of the Lefschetz number, which plays the main role in this paper. If we use such blocks also for other purposes (like in [Sr3]), that assumption is dropped.

Remark 2.2.5. Let φ be defined as the composition $\omega \circ \psi$ of a local flow ψ and a process ω (see Section 2.1) and let B be a block for ψ with the exit set B^- . If ω is T -periodic and

$$B = \omega_{(0, T)}(B), \quad B^- = \omega_{(0, T)}(B^-)$$

then it is not difficult to observe that

$$W = \{(t, x) \in \mathbb{R} \times X: \omega_{(0, t)}^{-1}(x) \in B\}$$

is a T -periodic block for φ (provided B and B^- are ENRs) and

$$W^- = \{(t, x) \in \mathbb{R} \times X: \omega_{(0, t)}^{-1}(x) \in B^-\}.$$

Various examples of blocks constructed in that way will be presented in the forthcoming sections.

Through reminder of this section W denotes a T -periodic block for φ . Define a map $\tau: W \longrightarrow [0, \infty]$ by

$$\tau(z) = \sup \{t > 0: \forall s \in [0, t]: \varphi_s^*(z) \in W\} \in [0, \infty],$$

The following property of W was essentially proved in [Wa].

Proposition 2.2.6. The map τ is continuous.

Proof: Denote

$$W^0 = \{z \in W: \exists t \geq 0: \varphi_t^*(z) \notin W\}.$$

Since W and W^- are closed in $\mathbb{R} \times X$, and, since W^0 is open in W , by the argument in [C1] or [C2, p.25], τ is continuous at each $z \in W^0$.

Let $z \in W$ and $\varphi_t^*(z) \in W$ for each $t \geq 0$ such that (z, t) belongs to the domain of φ^* . Then $\tau(z) = \infty$ by the compactness of $W \cap [0, s] \times X$ for every $s \geq 0$. Let $r > 0$. The set $\varphi^*({z} \times [0, r])$ is compact and does not intersect the closed set W^- , hence, if z' is sufficiently close to z , then also

$$\varphi^*({z'} \times [0, r]) \cap W^- = \emptyset.$$

If $z' \in W$ then $\varphi^*({z'} \times [0, r]) \subseteq W$, thus $\tau(z') \geq r$. ■

2.3. Proofs of the main theorems. We prove simultaneously both Theorems A and B stated in Section 1.2.

Let us fix a process ω as in the definition of the T -periodic proper pair (W, W^-) (see (iii) in Definition 2.2.2). Consider the topological direct sum of the spaces $W(0)$ and $W^-(0) \times S^1$. In that sum we identify each point $x \in W^-(0) \subseteq W(0)$ with $(x, 1) \in W^-(0) \times S^1$. Denote by C the obtained quotient space. By the assumptions, C is a compact ENR. We introduce a mapping

$$\gamma: C \times [0, 1] \longrightarrow C$$

as follows. Let $(z, t) \in C \times [0, 1]$. At first assume that

$z = (y, u) \in W^-(0) \times S^1$. Define

$$\gamma(z, t) = \gamma((y, u), t) = (\omega_{(0, T)}(y), u \exp((1-t)\pi i)) \in W^-(0) \times S^1.$$

Assume now that $z \in W(0)$. Let τ be the map in Proposition 2.2.6.

Consider two cases: $\tau(0, z) \leq (1-t)T$ and $\tau(0, z) \geq (1-t)T$. In the first case put

$$\gamma(z, t) = (\omega_{(\tau(0, z), T - \tau(0, z))}(\varphi_{(0, \tau(0, z))}(z)), \exp((1-t - \frac{\tau(0, z)}{T})\pi i)) \in W^-(0) \times S^1$$

and in the second case

$$\gamma(z, t) = \omega_{((1-t)T, tT)}(\varphi_{(0, (1-t)T)}(z)) \in W(0).$$

By Proposition 2.2.6 the map γ is continuous. In the sequel we will write γ_t instead of $\gamma(\cdot, t)$. Put

$$V = \{z \in W(0) : \tau(0, z) > T\}$$

If $z \in V$ then $\gamma_0(z) = \varphi_{(0, T)}(z)$. Recall that K_W denote the set of those fixed points of $\varphi_{(0, T)}$ that their trajectories (starting at the time 0) are wholly contained in W . Since $K_W \subseteq V$ and K_W is also equal to the set of fixed points of γ_0 ,

$$\text{ind}(\gamma_0, K_W) = \text{ind}(\gamma_0|_V, K_W) = \text{ind}(\varphi^\#, K_W),$$

where

$$\varphi^\#: V \ni z \longrightarrow \varphi_{(0, T)}(z) \in W(0).$$

In the case W is isolating we assert that K_W is compact and isolated in the set of fixed points of $\varphi_{(0, T)}$. The proof of that claim is postponed to the end of the current section (see Lemma 2.3.1 below). Thus we have

$$\text{ind}(\varphi^\#, K_W) = \text{ind}(\varphi_{(0, T)}, K_W).$$

In order to prove the theorems it suffices to compute $\text{ind}(\gamma_0, K_W)$.

By the homotopy property of the index and the Lefschetz Fixed Point Theorem (compare [D, Th. VII.6.6])

$$\text{ind}(\gamma_0, K_W) = \text{Lef}(\gamma_1).$$

γ_1 is a homeomorphism of the triad $(C; W(0), W^-(0) \times S^1)$ onto itself given by

$$\begin{aligned}\gamma_1(z) &= \omega_{(0,T)}(z) \text{ if } z \in W(0), \\ \gamma_1(y,u) &= (\omega_{(0,T)}(y), u) \text{ if } (y,u) \in W^-(0) \times S^1.\end{aligned}$$

Define

$$\begin{aligned}\Omega_1: W(0) \ni x &\longrightarrow \omega_{(0,T)}(x) \in W(0), \\ \Omega_2: W^-(0) \times S^1 \ni (y,u) &\longrightarrow (\omega_{(0,T)}(y), u) \in W^-(0) \times S^1, \\ \Omega_3: W^-(0) \ni x &\longrightarrow \omega_{(0,T)}(x) \in W^-(0).\end{aligned}$$

Since Ω_2 is homotopic to a map without fixed points, $\text{Lef}(\Omega_2) = 0$.

The Mayer-Vietoris exact sequence of the triad, the exact sequence of the pair $(W(0), W^-(0))$, and (2.4) imply that

$$\text{Lef}(\gamma_1) = \text{Lef}(\Omega_1) + \text{Lef}(\Omega_2) - \text{Lef}(\Omega_3) = \text{Lef}_T(W, W^-).$$

The proof is thus completed. ■

Lemma 2.3.1. If W is isolated then K_W is a compact isolated set of fixed points of the Poincaré map $\varphi_{(0,T)}$.

Proof: First we prove, that K_W is closed in X . Let $y \in \text{cl } K_W$ and b , $0 < b \leq \infty$, be such that

$$[0, b) = \{t \geq 0: \varphi_{(0,t)}(y) \text{ is defined}\}.$$

Thus $\varphi_s^*(0, y) \in W$ for every $s \in [0, b)$ (recall, that φ^* is defined by (2.1)). Since $W \cap [0, T] \times X$ is compact, $b > T$. Hence y is in the domain of $\varphi_{(0,T)}$ and thus $y \in K_W$. The set K_W is also compact, because it is contained in $W(0)$. If $x \in K_W \cap \partial W(0)$ then $\varphi_t^*(0, x) \in W$ for $t \in \mathbb{R}$, which is impossible, because W is isolating. The same argument shows, that if y is a fixed point of $\varphi_{(0,T)}$ then either

$\varphi_{(0,t)}(y) \in \text{int } W(t)$ for every $t \in [0, T]$ or there is an $s \in (0, T)$ such that $\varphi_{(0,s)}(y) \notin W(s)$, which implies that K_W is open in the set of fixed points of the Poincaré map. ■

2.4. Some corollaries. In this section we assume that φ is a local T -periodic process on an ENR-space X . The following result is an immediate consequence of (2.5) and Theorems A and B. The first claim in that result was established in [Sr2], the second one in slightly restricted form appeared in [Sr1].

Corollary 2.4.1. Let (C, D) be a pair of compact ENRs contained in X such that

$$D \times \mathbb{R} = (C \times \mathbb{R})^-.$$

If $\chi(C) - \chi(D)$ is different from zero, then there is a T -periodic trajectory of φ which is contained in $C \times \mathbb{R}$. If there are no trajectories of φ contained in $C \times \mathbb{R}$ which intersect $\partial C \times \mathbb{R}$ then

$$\text{ind}(\varphi_{(0,T)}, K_{C \times \mathbb{R}}) = \chi(C) - \chi(D). \quad \blacksquare$$

In Section 5.2 we apply the above proposition to a result in the theory of isolated invariant sets.

The other result presented here also relates to the Euler characteristic and is a corollary from Theorem A and its proof in Section 2.3.

Corollary 2.4.2. Let W be a T -periodic block for φ . If $W(0)$ is a compact and connected polyhedron, $W^-(0)$ is a subpolyhedron of

$W(0)$ and

$$\chi(W(0)) - \chi(W^-(0)) \neq 0,$$

then there is an $x \in W(0)$ such that $(0, x)$ is a subharmonic periodic point, i.e. $\varphi_{(0,T)}^k(x) = x$ for some positive integer k .

Proof: It suffices to show, that there is a k such that $\text{Lef}_{kT}(W, W^-) \neq 0$. Under the notation from the proof of Theorem A, $\text{Lef}_{kT}(W, W^-) = \text{Lef}(\gamma_1^k)$ for each k , γ_1 is a homeomorphism of C into itself and, by assumption, C is a connected polyhedron,

$$\chi(C) = \chi(W(0)) - \chi(W^-(0)) \neq 0.$$

The result follows from the proof of [DG, Proposition (4.6)] (that proposition is due to Fuller, [Fu]). ■

Chapter 3. Blocks for ordinary differential equations.

This chapter is devoted to the presentation of constructions of blocks for processes generated by differential equations. All those constructions will be applied in Part II. In Section 3.1 we present relations between processes and equations. A general method of construction of periodic blocks is presented in Section 3.2. Section 3.3 is devoted to presentation of blocks arising in the Floquet theory. Further investigations on construction of blocks are presented in Section 3.4 where it is assumed, that functions considered are homogeneous.

3.1. Processes generated by equations. In the sequel we consider an ordinary non-autonomous differential equation (1.1)

where $f: \mathbb{R} \times M \longrightarrow TM$ is a time-dependent continuous vector-field on a smooth (i.e. of the class C^∞) Riemannian manifold M . In this section we assume that the Cauchy problem associated to (1.1) has the uniqueness property (it suffices to assume, for example, that f is of the class C^1).

The equation (1.1) generates a local process as follows. If $x_0 \in M$, $t_0 \in \mathbb{R}$ and $x(t_0, x_0; \cdot)$ is the saturated solution of (1.1) such that $x(t_0, x_0; t_0) = x_0$ (i.e. the solution through (t_0, x_0)), then we put

$$\varphi_{(t_0, \tau)}(x_0) = x(t_0, x_0; t_0 + \tau).$$

If f is T -periodic with respect to t then φ is a T -periodic local process. The trajectory of (σ, x) in φ is just equal to the graph of solution of (1.1) through (σ, x) .

In particular, if the equation

$$(3.1) \quad \dot{x} = A(t)x,$$

where A is a continuous real- $n \times n$ -matrix valued function, has the fundamental matrix $\Phi(t)$ for which $\Phi(0) = I$, then it generates a process φ on \mathbb{R}^n given by

$$(3.2) \quad \varphi_{(\sigma, t)} = \Phi(\sigma + t)\Phi(\sigma)^{-1}.$$

If φ is generated by (1.1) then the local flow φ^* (defined by (2.1)) is generated by the equation

$$(\dot{t}, \dot{x}) = (1, f(t, x))$$

on $\mathbb{R} \times M$. Let local processes ψ and ω be generated by

$$(3.3) \quad \dot{x} = g(t, x),$$

$$(3.4) \quad \dot{x} = h(t, x)$$

respectively. By simple calculations we conclude, that their

composition $\omega \circ \psi$ (compare (2.2)) is in that case generated by the equation

$$(3.5) \quad \dot{x} = h(t, x) + d\omega_{(0, t)}(\omega_{(0, t)}^{-1}(x))(g(t, \omega_{(0, t)}^{-1}(x))).$$

In particular, if $h(t, x) = H(t)x$ for some matrix-valued function H with the fundamental matrix $\Omega(t)$, $\Omega(0) = I$, then

$$(\omega \circ \psi)_{(\sigma, t)}(x) = \Omega(\sigma + t)\psi_{(\sigma, t)}(\Omega(\sigma)^{-1}x)$$

and (3.5) becomes

$$(3.6) \quad \dot{x} = H(t)x + \Omega(t)g(t, \Omega(t)^{-1}x).$$

3.2. Strong and weak periodic blocks. We still consider the equation (1.1) on the Riemannian manifold M . Unless otherwise stated, we only assume that the vector-field f in the right-hand side of (1.1) is continuous, in particular we do not assume that the equation generates a local process. We modify the notion of T -periodic block in order to obtain an analogous notion in the most convenient form for differential equations. In fact, we introduce two forms of such blocks - strong and weak.

Let W be a subset of $\mathbb{R} \times M$ and T be a positive number.

Definition 3.2.1. W is called a strong T -periodic block for the equation (1.1) (or, equivalently, for the vector-field f) provided there exist continuous functions $L^1, \dots, L^s: M \longrightarrow \mathbb{R}$, for some $s \geq 1$, a continuous time-dependent vector-field $h: \mathbb{R} \times M \longrightarrow TM$ generating a process ω on M , such that

$$W = \{(t, x) \in \mathbb{R} \times M: \forall i = 1, \dots, s: \Lambda^i(t, x) \leq 0\},$$

where $\Lambda^i: \mathbb{R} \times M \longrightarrow \mathbb{R}$ is defined by

$$\Lambda^i(t, x) = L^i(\omega_{(0, t)}^{-1}(x)),$$

and the following conditions are satisfied:

(A1) the set

$$C = \{x \in M: \forall i = 1, \dots, s: L^i(x) \leq 0\},$$

is compact,

(A2) L^i is of the class C^1 in some neighborhood of the set $\{x \in C: L^i(x) = 0\}$ for each $i = 1, \dots, s$,

(A3) For every $x \in C$ the set $\{\text{grad } L^i(x): i \text{ such that } L^i(x) = 0\}$ is linearly independent,

(A4) Λ^i is T-periodic in the first variable, $i = 1, \dots, s$,
and the inequalities:

$$(3.7) \quad \text{grad } \Lambda^i(t, x) \cdot (1, f(t, x)) > 0 \quad (\forall i = 1, \dots, r, (t, x) \in W, \Lambda^i(t, x) = 0),$$

$$(3.8) \quad \text{grad } \Lambda^i(t, x) \cdot (1, f(t, x)) < 0 \quad (\forall i = r+1, \dots, s, (t, x) \in W, \Lambda^i(t, x) = 0)$$

for some r , $0 \leq r \leq s$, are satisfied.

W is called a weak T-periodic block for (1.1) (or: for f) provided all the above conditions are satisfied except the last inequalities; (3.7), (3.8) are replaced by:

$$\begin{aligned} \text{grad } \Lambda^i(t, x) \cdot (1, f(t, x)) &\geq 0 \quad (\forall i = 1, \dots, r, (t, x) \in W, \Lambda^i(t, x) = 0) \\ \text{grad } \Lambda^i(t, x) \cdot (1, f(t, x)) &\leq 0 \quad (\forall i = r+1, \dots, s, (t, x) \in W, \Lambda^i(t, x) = 0) \end{aligned}$$

respectively.

Under the above notation, the set

$$W^- = \{(t, x) \in W: \exists i = 1, \dots, r: \Lambda^i(t, x) = 0\}$$

is called the exit set of a strong (or weak) T-periodic block W .

A relation to the previous notion of T-periodic block (see Definition 2.2.4) is established in the following result:

Proposition 3.2.2. If W is a strong T-periodic block for (1.1) and f generates a local process φ then W is also a T-periodic isolating block for φ and their exit sets with respect to φ and to (1.1) coincide.

Proof. The inequalities (3.7), (3.8) imply the required form of the exit set W^- for φ . It follows by (A2), (A3) that the set C is a manifold-with-corners, hence W is also such a manifold. By a similar argument W^- is a topological manifold with boundary. In particular, both the sets are ENRs. This observation together with (A1) and (A4) imply that (W, W^-) is a T-periodic proper pair of ENRs, hence the result follows. ■

Remark 3.2.3. Assume the notation used in Definition 3.2.1.

Put

$$D = \{x \in C: \exists i = 1, \dots, r: L^i(x) = 0\}.$$

Then

$$W^- = \{(t, x) \in \mathbb{R} \times M: x \in \omega_{(0, T)}(D)\}.$$

It follows by (2.4) that

$$(3.9) \quad \text{Lef}_T(W, W^-) = \text{Lef}(\omega_{(0, T)}|_C) - \text{Lef}(\omega_{(0, T)}|_D).$$

In practice it is not difficult to construct sets which

fulfill all the required properties of a strong T -periodic block for (1.1) except for the inequalities (3.7), (3.8). In the rest of this section (and also in the forthcoming Section 3.4) we present constructions which lead to these inequalities. Actually our argument will go in the opposite direction: we will match the vector-field f to a given set in order to obtain the desired inequalities.

As in Definition 3.2.1 let us consider two integers r and s , $s \geq 1$, $0 \leq r \leq s$, continuous functions $L^1, \dots, L^s: M \longrightarrow \mathbb{R}$ and the set C consisting of points $x \in M$ such that $L^i(x) \leq 0$ for each $i = 1, \dots, s$. Let $g: \mathbb{R} \times M \longrightarrow TM$ be a continuous vector-field and assume that the inequalities

$$(3.10) \quad \text{grad } L^i(x) \cdot g(t, x) > 0 \quad (\forall i=1, \dots, r, (t, x) \in \mathbb{R} \times C, \\ L^i(x) = 0),$$

$$(3.11) \quad \text{grad } L^i(x) \cdot g(t, x) < 0 \quad (\forall i = r+1, \dots, s, (t, x) \in \mathbb{R} \times C, \\ L^i(x) = 0)$$

are satisfied. In order to present an example in which such a situation takes place, let us assume that g is C^∞ and does not depend of time. It is proved in [WY] that in any neighborhood of an isolated invariant set for the local flow determined by g there exists an isolating block having the form of the set C satisfying the inequalities (3.10), (3.11) and the conditions (A2), (A3). The proof in [WY] uses smooth Lyapunov functions constructed in [Wi]. An improved construction of those functions is presented in [N].

Let $h: \mathbb{R} \times M \longrightarrow TM$ be a time-dependent continuous vector-field, generating a process ω on M . For $i = 1, \dots, s$ put

$$\Lambda^i: \mathbb{R} \times M \ni (t, x) \longrightarrow L^i(\omega_{(0,t)}^{-1}(x)) \in \mathbb{R}.$$

Define

$$W = \{(t, x) \in \mathbb{R} \times M: \forall i = 1, \dots, s: \Lambda^i(t, x) \leq 0\}$$

We assume the following two conditions:

(A5) g and h are T -periodic in the first variable,

(A6) ∂C is compact.

(Note, that $\partial C = \{x \in C: \exists i = 1, \dots, s: L^i(x) = 0\}$ provided (A3) is satisfied.) Let c be a positive number such that

$$(3.12) \quad \|\text{grad } L^i(x)\| < c \quad (\forall i = 1, \dots, s, x \in C, L^i(x) = 0).$$

Its existence follows by (A6). It follows by (3.10), (3.11) and

(A5), (A6) that we can choose a constant a such that

$$(3.13) \quad |\text{grad } L^i(x) \cdot g(t, x)| > a \quad (\forall i = 1, \dots, s, (t, x) \in \mathbb{R} \times C, L^i(x) = 0).$$

Put

$$d = \max \{\|\text{d}\omega_{(0,t)}^{-1}(\omega_{(0,t)}(y))\|: y \in \partial C, t \in [0, T]\}.$$

Proposition 3.2.4. Assume the conditions (A2) - (A6). Let $p: \mathbb{R} \times M \longrightarrow TM$ be a continuous time-dependent vector-field on M . Let $f(t, x)$ be equal to the right-hand side of (3.5) perturbed by $p(t, x)$, i.e.

$$f(t, x) = h(t, x) + \text{d}\omega_{(0,t)}(\omega_{(0,t)}^{-1}(x))(g(t, \omega_{(0,t)}^{-1}(x))) + p(t, x).$$

Assume that f is T -periodic in t . If for each $(t, x) \in \partial W$

$$\|p(t, x)\| < a/(cd)$$

then the inequalities (3.7), (3.8) are satisfied.

Proof: Let $(t, x) \in W$, $i = 1, \dots, r$ and $\Lambda^i(t, x) = 0$. The

T-periodicity in t of f and Λ^i (compare (A4)) imply that if $t = kT + s$ for some $k \in \mathbb{Z}$, $s \in [0, T)$ then

$$\text{grad } \Lambda^i(t, x) \cdot (1, f(t, x)) = \text{grad } \Lambda^i(s, x) \cdot (1, f(s, x)),$$

hence, in order to prove the inequalities, we can assume that $t \in [0, T)$. The vector $(1, h(t, x))$ is tangent at (t, x) to the zero-level surface of Λ^i (because, by definition, Λ^i is constant along the trajectories of ω), so (3.12), (3.13), the Schwartz Lemma and the choice of d imply that

$$\begin{aligned} & \text{grad } \Lambda^i(t, x) \cdot (1, f(t, x)) \\ &= \text{grad } L^i(\omega_{(0,t)}^{-1}(x)) \cdot (g(t, \omega_{(0,t)}^{-1}(x)) + d\omega_{(0,t)}^{-1}(x)(p(t, x))) \\ &> a - \|\text{grad } L^i(\omega_{(0,t)}^{-1}(x))\| \|d\omega_{(0,t)}^{-1}(x)\| \|p(t, x)\| > 0. \end{aligned}$$

The same argument applied to the case $i = r+1, \dots, s$ leads to the second inequality, so the result follows. ■

3.3. Periodic blocks in the Floquet theory. The analysis of the standard proof of the Floquet Theorem (compare [CL, Ch.3, Sec.5] delivers natural examples of periodic blocks constructed by the method introduced in the previous section.

Consider the equation (3.1) in which A is a continuous T -periodic function on the space of real $n \times n$ -matrices. Let $\Phi(t)$ be the fundamental matrix of (3.1), $\Phi(0) = I$. Denote by μ_1, \dots, μ_n the characteristic multipliers (i.e. the eigenvalues of $\Phi(T)$). Assume that $|\mu_i| \neq 1$ for each i . Let m_1, m_2, m_3 be nonnegative integers, $m_1 + m_2 + m_3 \leq n$ and suppose that

$$M_1 = \{\mu_1, \dots, \mu_{m_1}\} \subseteq (-\infty, -1),$$

$$M_2 = \{\mu_{m_1+1}, \dots, \mu_{m_1+m_2}\} \subseteq \{z \in \mathbb{C} : |z| > 1, z \notin (-\infty, -1)\},$$

$$M_3 = \{\mu_{m_1+m_2+1}, \dots, \mu_{m_1+m_2+m_3}\} \subseteq \{z \in \mathbb{C}: |z| < 1, z \notin (-1, 0)\},$$

$$M_4 = \{\mu_{m_1+m_2+m_3+1}, \dots, \mu_n\} \subseteq (-1, 0).$$

Let V_i denotes the subspace of \mathbb{R}^n spanned by the generalized eigenspaces of M_i , $i = 1, \dots, 4$. (Put $V_1 = \{0\}$ in the case $M_1 = \emptyset$, i.e. $m_1 = 0$.) Let $G_i: V_i \longrightarrow V_i$ be a linear operator such that

$$\exp(G_i T) = \Phi(T)|_{V_i} \text{ for } i = 2, 3,$$

$$\exp(G_i T) = -\Phi(T)|_{V_i} \text{ for } i = 1, 4.$$

(See [CL, Ch.3, Problems 40,41], and also [CL, Ch.3, Sec.1], for the construction of G_i .) Assume that $L^i_\#: V_i \longrightarrow \mathbb{R}$ is a quadratic form being simultaneously a Lyapunov function (from the usual proof of the Lyapunov Theorem) for

$$\dot{x} = G_i x$$

if $i = 3, 4$ and for

$$\dot{x} = -G_i x$$

if $i = 1, 2$. (If $V_i = \{0\}$ put $L^i_\# = 0$.) For $i = 1, \dots, 4$ define

$L^i: \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$L^i(x) = L^i_\#(x_i) - 1,$$

where $x = x_1 + x_2 + x_3 + x_4$ and $x_j \in V_j$. Put

$$B = \{x \in \mathbb{R}^n: L^i(x) \leq 0, i = 1, \dots, 4\}.$$

By G denote the direct sum of G_i , $i = 1, \dots, 4$. It follows that

$$\text{grad } L^i(x) \cdot Gx > 0 \text{ (} i=1, 2, x \in B, L^i(x) = 0 \text{),}$$

$$\text{grad } L^i(x) \cdot Gx < 0 \text{ (} i=3, 4, x \in B, L^i(x) = 0 \text{),}$$

hence B is an isolating block for the flow generated by G , and

$$B^- = \{x \in B: L^i(x) = 0, i = 1 \text{ or } i = 2\}.$$

Put

$$\Omega(t) = \Phi(t) \exp(-Gt).$$

By a direct calculation we assert that $\Omega(t)$ is a fundamental matrix of the equation

$$\dot{x} = H(t)x,$$

where

$$H(t) = A(t) - \Omega(t) G \Omega(t)^{-1}.$$

Moreover, $\Omega(0) = I$ and another calculation shows that H is T -periodic (although, in general, Ω is only $2T$ -periodic, as one can compute directly). Indeed, by the T -periodicity of A and the commutativity of G with $\Phi(T)$ we obtain

$$\begin{aligned} \Omega(t+T) G \Omega(t+T)^{-1} &= \Phi(t) \Phi(T) \exp(-G(t+T)) G \exp(G(t+T)) \Phi(T)^{-1} \Phi(t)^{-1} \\ &= \Phi(t) G \Phi(t)^{-1} = \Omega(t) G \Omega(t)^{-1}. \end{aligned}$$

For $i = 1, \dots, 4$ put

$$(3.14) \quad \Lambda^i(t, x) = L^i(\Omega(t)^{-1}x).$$

We assert that Λ^i is T -periodic in t . Indeed, by T -periodicity of H ,

$$\Lambda^i(t+T, x) = L^i(\Omega(t+T)^{-1}x) = L^i(\Omega(T)^{-1}\Omega(t)^{-1}x).$$

By the definitions of G , Ω , and L ,

$$\begin{aligned} L^i(\Omega(T)^{-1}\Omega(t)^{-1}x) &= L_{\#}^i((- (\Omega(t)^{-1}x)_1 + (\Omega(t)^{-1}x)_2 + (\Omega(t)^{-1}x)_3 - (\Omega(t)^{-1}x)_4)_1) - 1 \\ &= L_{\#}^i(\pm (\Omega(t)^{-1}x)_1) - 1. \end{aligned}$$

Since $L_{\#}^i$ is a quadratic form, the assertion follows.

We apply Proposition 3.2.4 with $f(t, x) = A(t)x$,

$h(t, x) = H(t)x$, $g(t, x) = Gx$, and $p \equiv 0$. As a conclusion we get the following result:

Proposition 3.3.1. The set

$$W = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \Lambda^i(t, x) \leq 0, i = 1, \dots, 4\}$$

is a strong T -periodic isolating block for (3.1) having

$$W^- = \{(t, x) \in W : \Lambda^i(t, x) = 0, i = 1 \text{ or } 2\}$$

as the exit set. ■

In fact, the periodic block W was constructed by the method proposed in Remark 2.2.5.

Remark 3.3.2. We determine the associated Lefschetz numbers (see Definition 2.2.3)) as follows. The set B is homeomorphic to a ball, thus for each $k \in \mathbb{Z}$,

$$\text{Lef}(\Omega(kT)|_B) = 1.$$

The set B^- is homeomorphic to

$$S^{m_1-1} \times B^{m_2} \times B^{m_3} \times B^{m_4} \cup B^{m_1} \times S^{m_2-1} \times B^{m_3} \times B^{m_4},$$

hence it has the homotopy type of (m_1+m_2-1) -dimensional sphere. If k is even, then $\Omega(kT)$ is equal to the identity, thus

$$\text{Lef}(\Omega(kT)|_{B^-}) = \chi(S^{m_1+m_2-1}) = 1 - (-1)^{m_1+m_2}.$$

If k is odd, then $\Omega(kT)|_{B^-}$ behaves like the map

$$S^{m_1-1} \times B^{m_2} \cup B^{m_1} \times S^{m_2-1} \ni (x, y) \longrightarrow (-x, y) \in S^{m_1-1} \times B^{m_2} \cup B^{m_1} \times S^{m_2-1},$$

hence

$$\text{Lef}(\Omega(kT)|_{B^-}) = 1 + (-1)^{m_1+m_2-1} (-1)^{m_1} = 1 - (-1)^{m_2}.$$

Using the formula (3.9) we compute $\text{Lef}_{kT}(W, W^-)$ for various k . For the reasons explained below we change slightly our notation. Put

$m = m_1 + m_2$ and $l = m_1$. Then

$$\text{Lef}_{kT}(W, W^-) = \begin{cases} (-1)^{m-1} & \text{if } k \text{ is odd} \\ (-1)^m & \text{if } k \text{ is even} \end{cases}$$

The role of the above numbers l and m was indicated in a different context in the paper [MR]. From that paper we borrow the following terminology:

Definition 3.3.3. A T -periodic continuous real matrix-valued function A is called hyperbolic if the characteristic multipliers of (2.3) are outside of the unit circle. In that case the pair (m, l) is called the Morse index of A if m is the number of the multipliers outside of the unit disc and l is the number of the multipliers in $(-\infty, -1)$ (each multiplier is counted with its multiplicity).

Obviously, if A is constant, then the above number l is even for each T .

3.4. Homogeneous equations. In this section we assume that $M = \mathbb{R}^n \setminus 0$ and q is a real number. A function $k: \mathbb{R}^n \setminus 0 \longrightarrow \mathbb{R}^m$ is called homogeneous of degree q if $k(\lambda x) = \lambda^q k(x)$ for every $\lambda > 0$ and every $x \in \mathbb{R}^n \setminus 0$.

We use the same notation as in Section 3.2, however we assume here that h is linear in the second variable, i.e.

$$h(t, x) = H(t)x$$

for some matrix-valued function H . By $\Omega(t)$ we denote the

fundamental matrix of the equation (3.4) such that $\Omega(0) = I$. It follows in particular that the function Λ^1 is given by (3.14) and

$$W = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \Omega(t)^{-1}x \in C\}.$$

For every $\varepsilon > 0$ and every $i = 1, \dots, s$ define

$$W_\varepsilon = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : (t, \varepsilon^{-1}x) \in W\},$$

$$\Lambda_\varepsilon^i(t, x) = \Lambda(t, \varepsilon^{-1}x),$$

hence

$$\Lambda_\varepsilon^1(t, x) = L^1(\varepsilon^{-1}\Omega(t)^{-1}x),$$

$$W_\varepsilon = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \forall i = 1, \dots, s: \Lambda_\varepsilon^i(t, x) \leq 0\}$$

$$= \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : x \in \Omega(t)(\varepsilon C)\},$$

$$W_1 = W.$$

Let p be continuous map $\mathbb{R} \times (\mathbb{R}^n \setminus 0) \longrightarrow \mathbb{R}^n$. We consider the following two conditions imposed on p :

$$(B1) \quad p(t, x)/\|x\|^q \longrightarrow 0 \text{ as } \|x\| \longrightarrow \infty \text{ uniformly in } t \in \mathbb{R}.$$

$$(B2) \quad p(t, x)/\|x\|^q \longrightarrow 0 \text{ as } \|x\| \longrightarrow 0 \text{ uniformly in } t \in \mathbb{R}.$$

Proposition 3.4.1. Assume that the conditions (A2) - (A6) and the inequalities (3.10), (3.11) are satisfied. Assume moreover that $0 \notin \partial C$, f is T -periodic in the first variable and g is homogeneous of degree q with respect to the second variable. Then there exist positive real numbers ε_0 and ε_∞ such that the inequalities (3.7), (3.8) are satisfied with W and Λ replaced by W_ε and Λ_ε , respectively, provided one of the following conditions holds:

$$(i) \quad q > 1, (B1) \text{ is satisfied, } \varepsilon > \varepsilon_\infty \text{ and}$$

$$(3.15) \quad f(t, x) = \Omega(t)g(t, \Omega(t)^{-1}x) + p(t, x).$$

$$(ii) \quad q \leq 1, (B1) \text{ is satisfied, } \varepsilon > \varepsilon_\infty \text{ and}$$

$$(3.16) \quad f(t, x) = H(t)x + \Omega(t)g(t, \Omega(t)^{-1}x) + p(t, x).$$

(iii) $q \geq 1$, (B2) is satisfied, $0 < \varepsilon < \varepsilon_0$ and f is given by (3.16).

(iv) $q < 1$, (B2) is satisfied, $0 < \varepsilon < \varepsilon_0$ and f is given by (3.15).

Proof. For $i = 1, \dots, s$ and $\varepsilon > 0$ put $L_\varepsilon^i(x) = L^i(\varepsilon^{-1}x)$, hence

$$\Lambda_\varepsilon^i(t, x) = L_\varepsilon^i(\Omega(t)^{-1}x),$$

$$\varepsilon C = \{x \in \mathbb{R}^n: \forall i=1, \dots, s: L_\varepsilon^i(x) \leq 0\},$$

and for every $x \in \varepsilon C$ such that $L_\varepsilon^i(x) = 0$:

$$\text{grad } L_\varepsilon^i(x) \cdot g(t, x) = \varepsilon^{q-1} \text{grad } L^i(\varepsilon^{-1}x) \cdot g(t, \varepsilon^{-1}x).$$

Let positive numbers a and c fulfill the conditions (3.13) and (3.12) respectively. Then

$$|\text{grad } L_\varepsilon^i(x) \cdot g(t, x)| > \varepsilon^{q-1}a, \quad \|\text{grad } L_\varepsilon^i(x)\| < \varepsilon^{-1}c.$$

for every $i = 1, \dots, s$ and $x \in \varepsilon C$ such that $L_\varepsilon^i(x) = 0$. In our case the constant d introduced in Section 3.2 is given by

$$d = \max \{\|\Omega(t)^{-1}\|: t \in [0, T]\}.$$

By Proposition 3.2.4 the inequalities (3.7), (3.8) with W replaced by W_ε , Λ replaced by Λ_ε , and f given by

$$f(t, x) = H(t)x + \Omega(t)g(t, \Omega(t)^{-1}x) + r(t, x)$$

(for a suitable function r) are satisfied provided

$$\|r(t, x)\| < \varepsilon^q a / (cd)$$

for every $(t, x) \in \partial W_\varepsilon$.

Choose positive numbers ρ_1 and ρ_2 such that

$$\{x \in \mathbb{R}^n: \exists t \in \mathbb{R}: (t, x) \in \partial W\} \subseteq \{x \in \mathbb{R}^n: \rho_1 < \|x\| < \rho_2\}.$$

The existence of such numbers follows from the assumptions

(A4), (A6) guaranteeing the compactness of the image of ∂W under the projection onto \mathbb{R}^n , and the assumption on ∂C which guarantees that no trajectory of ω through a point in $\{0\} \times \partial C$ intersects the time-axis.

Assume that the condition (i) holds. By (B1), the choice of q , linearity and periodicity of $H(t)$, we conclude the existence of a positive number R such that

$$\|p(t, x) - H(t)x\| < (a/cd\rho_2^q)\|x\|^q$$

if $\|x\| > R$ and $t \in \mathbb{R}$. Let ε_∞ fulfill $\varepsilon_\infty \rho_1 > R$. If $\varepsilon > \varepsilon_\infty$ and $(t, x) \in \partial W_\varepsilon$ then $(t, \varepsilon^{-1}x) \in \partial W$, hence

$$R < \varepsilon \rho_1 < \|x\| < \varepsilon \rho_2,$$

hence

$$\|p(t, x) - H(t)x\| < a\varepsilon^q \rho_2^q / (cd\rho_2^q).$$

Thus, if we put $r(t, x) = p(t, x) - H(t)x$, we complete the proof in the case (i).

One can prove (ii) by exactly the same argument, remembering that here $H(t)x$ has no longer lower growth at infinity than $\|x\|^q$. Thus (3.15) is replaced by (3.16) in the conclusion. Proofs of (iii) and (iv) can be done by the analogous argument and are left to the reader. ■

Remark 3.4.2. The set W_ε is a strong T -periodic block for (1.1) provided (A1) and the assumptions of the above proposition are satisfied. Put

$$Y = \{(t, x) \in W : \exists i = 1, \dots, r: \Lambda^i(t, x) = 0\}.$$

Under the notation used in Remark 3.2.3, we have

$$W_{\varepsilon}^{-} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : x \in \Omega(t)(\varepsilon D)\} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : (t, \varepsilon^{-1}x) \in Y\}$$

and, by (3.9),

$$(3.17) \quad \text{Lef}_T(W_{\varepsilon}, W_{\varepsilon}^{-}) = \text{Lef}_T(W, Y) = \text{Lef}(\Omega(T)|_C) - \text{Lef}(\Omega(T)|_D).$$

Chapter 4. The periodic problem.

In this chapter we deal with the periodic problem for ordinary differential equations. In Section 4.1 we recall classical methods for that problem, like the method of guiding functions and the Mawhin's method. In Section 4.2 we present the geometric method in a form convenient for applications.

4.1. Usual methods. Throughout remainder of this chapter we consider the equation (1.1) on a Riemannian manifold M . We assume that f is continuous and T -periodic with respect to the first variable, where $T > 0$.

We recall briefly some usual methods for the periodic problem (1.1), (1.2) in the general case. We do not try to present those methods in the most general form, in particular we consider only the case $M = \mathbb{R}^n$. Below we assume that G is an open and bounded subset of \mathbb{R}^n . We begin with the following Theorem of Krasnosel'skii:

Theorem 4.1.1 ([K], [KZ, Lem.13.2]). If (1.1) satisfies the uniqueness and global existence of the associated Cauchy problem, there are no solutions of (1.1) such that $x(0) = x(s) \in \partial G$ for each $s \in (0, T]$, $f(0, x) \neq 0$ for every $x \in \partial G$ and

$$\deg(0, f(0, \cdot), G) \neq 0$$

then (1.1), (1.2) has a solution for which $x(0) \in G$.

The above theorem can be used in proofs of results concerning the method of guiding functions given by M.A. Krasnosel'skii and his collaborators. A C^1 -map $V: \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a guiding function for (1.1) provided there exists an $r > 0$ such that

$$(4.1) \quad \text{grad } V(x) \cdot f(t, x) > 0$$

for each $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $\|x\| \geq r$.

Theorem 4.1.2 ([KZ, Th.13.4], [RM, Th.5.4.3]). If there exists $m+1$ guiding functions V_0, V_1, \dots, V_m such that

$$\sum_{0 \leq j \leq m} |V_j(x)| \longrightarrow \infty \text{ as } \|x\| \longrightarrow \infty$$

and

$$\deg(0, \text{grad } V_0, G) \neq 0,$$

where G contains the ball centered at the origin having the radius r_0 (r_0 is associated to V_0 from the definition of a guiding function), then (1.1), (1.2) has a solution.

The other approach to the periodic problem was given by J. Mawhin. It applies the Leray-Schauder degree (and its modification, the coincidence degree). Below we present the basic result for his method. Let $F: \mathbb{R} \times \mathbb{R}^n \times [0, 1] \longrightarrow \mathbb{R}^n$ be a continuous map such that

$$f(t, x) = F(t, x, 1).$$

Theorem 4.1.3 ([RM, Th.5.3.17], compare also [Ma, Th.IV.13]).

Assume that for every $\lambda \in (0,1]$ the problem (1.2) for

$$\dot{x} = \lambda F(t, x, \lambda)$$

has no solution in $cl\ G$ which intersects ∂G . Assume moreover, that

$f^\# : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, averaged field of $F(\cdot, \cdot, 0)$, defined by

$$f^\#(x) = \frac{1}{T} \int_0^T F(t, x, 0) dt$$

has no zeros in ∂G . If

$$\deg(0, f^\#, G) \neq 0$$

then (1.1), (1.2) has a solution in G .

The above result is also an example of continuation theorems.

These theorems are based on carrying of the given problem to a simpler one by admissible homotopy. In the following continuation theorem the terminal equation of such a homotopy is autonomous.

Theorem 4.1.4 ([CMZ, Th.2]) Let a map F be as above. Assume that $F(\cdot, \cdot, 0)$ does not depend on the first variable and put

$$f_0(x) = F(t, x, 0).$$

Assume moreover, that for each $\lambda \in [0,1]$ the problem (1.2) for

$$\dot{x} = F(t, x, \lambda)$$

has no solution in $cl\ G$ which intersects ∂G . If

$$\deg(0, f_0, G) \neq 0$$

then (1.1), (1.2) has a solution in G .

In the above theorems 4.1.1, 4.1.3, and 4.1.4 it is assumed that there are no solutions of some periodic problem which are

contained in the closure of G and intersect its boundary. Frequently it is difficult to check that assumption, however it is always valid for so-called strict bound sets (see [Ma, Def.VII.1]). Roughly speaking, for each boundary point z of a strict bound set there is a function V defined in some neighborhood of z and satisfying (4.1) for each t . Results on strict bound sets are presented in [Ma, Ch.VIII].

4.2. The geometric method in differential equations. By Proposition 3.2.2, Theorems A and C can be directly applied to strong periodic blocks for (1.1), provided the Cauchy problem for (1.1) has the uniqueness property. Below we present how the theorems can be further improved, in particular we drop the uniqueness assumption. Recall that the notion of a weak periodic block was introduced in Definition 3.2.1.

Theorem 4.2.1. Let W be a weak T -periodic block for (1.1). If

$$\text{Lef}_T(W, W^-) \neq 0$$

then the problem (1.1), (1.2) has a solution whose graph is contained in W .

Theorem 4.2.2. Let W_1 and W_2 be weak T -periodic blocks for (1.1). If $W_1 \subseteq W_2$ and

$$\text{Lef}_T(W_1, W_1^-) \neq \text{Lef}_T(W_2, W_2^-)$$

then the problem (1.1), (1.2) has a solution whose graph is contained in W_2 and is not contained in $\text{int } W_1$.

Proof. Obviously, Theorem 4.2.2 is more general than 4.2.1 (put $W_1 = \emptyset$). In order to prove Theorem 4.2.2 it suffices to find a sequence of smooth functions approximating f , for which W_1 and W_2 are strong T -periodic blocks, and apply Proposition 3.2.2 and Theorem C to them. Each cluster function of the resulting sequence of solutions is a required solution of the problem. The existence of the approximating sequence of functions follows by [Ha, Sec.2] and the assumption (A3) in the definition of strong block. ■

Let us mention, that in order to apply Theorems 4.2.1 and 4.2.2, one does not need to consider solutions of any auxiliary periodic problem, like in Theorems 4.1.1, 4.1.3 and 4.1.4.

PART II. APPLICATIONS OF THE GEOMETRIC METHOD.

Chapter 5. General equations of first order.

We investigate two different topics. In Section 5.1 we consider perturbed linear equations and prove some existence results for periodic solutions. In Section 5.2 we present an application of the geometric method to the theory of isolated invariant sets.

5.1. Results concerning linear equations. We consider the problem of existence of periodic solutions of perturbed linear equations. Many papers are devoted to that topic, for a bibliography of related results the reader may consult [RM] or [RSC2]. Our main purpose is to present how proofs of some results on the problem can be based on the periodic blocks constructed in Section 3.3 and applications of Theorems 4.2.1 and 4.2.2. First of the results follows from a theorem in [LO] (compare also [RM, Cor.4.1.11, Lem.4.6.4]). Recall that hyperbolic matrix-valued functions and their Morse indices were introduced in Definition 3.3.3.

Proposition 5.1.1 (compare [LO, Th.1]). Let A be a hyperbolic T -periodic matrix-valued function and let $b: \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous function, globally bounded and T -periodic in the first variable. Then the equation

$$(5.1) \quad \dot{x} = A(t)x + b(t, x)$$

has a T-periodic solution.

Proof: Let W denote the strong T-periodic block for A considered in Proposition 3.3.1. Recall, that the construction in Section 3.3 leads to that form of periodic blocks which is required in Proposition 3.4.1 (with $q = 1$). By the assumed boundedness, the condition (B1) from Section 3.4 is satisfied for b . By Proposition 3.4.1(ii) we conclude the existence of a large $\epsilon > 0$ such that W_ϵ is a strong T-periodic block for (5.1). By Remarks 3.3.2 and 3.4.2, the associated Lefschetz number is equal to ± 1 , hence Theorem 4.2.1 implies the result. ■

The second result is rather well-known, however we do not indicate a single reference, in which it is established (see [MR] for a related result).

Proposition 5.1.2. Let A_0 and A_∞ be hyperbolic T-periodic matrix-valued functions with the Morse indices (m_0, l_0) and (m_∞, l_∞) respectively. Let $f: \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a T-periodic continuous function and

$$(f(t, x) - A_0(t)x) / \|x\| \longrightarrow 0 \text{ as } \|x\| \longrightarrow 0, \ x \neq 0$$

uniformly in t ,

$$(f(t, x) - A_\infty(t)x) / \|x\| \longrightarrow 0 \text{ as } \|x\| \longrightarrow \infty \text{ uniformly in } t.$$

Under the above assumptions:

- (i) There is a nontrivial T-periodic solution of (1.1),
provided $m_0 - m_\infty - l_0 + l_\infty$ is odd.

(ii) There is a nontrivial $2T$ -periodic solution of (1.1), provided $m_0 - m_\infty$ is odd.

Proof. Let W_0 and W_∞ denote the strong T -periodic blocks for A_0 and A_∞ , respectively, considered in Proposition 3.3.1. By Y_0 and Y_∞ denote its exit sets (with respect to A_0 and, respectively, A_∞). By Proposition 3.4.1(ii), (iii) we can find two numbers ε and η , $0 < \varepsilon < \eta$, such that $(W_0)_\varepsilon$ and $(W_\infty)_\eta$ are T -periodic blocks for the local process generated by the equation (1.1) and

$$(W_0)_\varepsilon \subseteq (W_\infty)_\eta.$$

In the case (i), by Remark 3.3.2,

$$\text{Lef}_T(W_0, Y_0) = (-1)^{m_0 - 1} \neq (-1)^{m_\infty - 1} = \text{Lef}_T(W_\infty, Y_\infty)$$

hence the conclusion follows by Theorem 4.2.2 and Remark 3.4.2.

The case (ii) follows by a similar argument. ■

Below we present another proposition. It seems to be more original than the previous ones. Its proof contains another construction of periodic blocks.

Through the remainder of this section we use the following notation. By n we denote a positive integer. For $i = 1, \dots, n$, let δ_i denote a nonzero real number and let $p_i: \mathbb{R} \longrightarrow \mathbb{R}$ denote a continuous T -periodic function such that $p_i(t) > 0$ for all i and t . Put

$$\Delta = \text{diag}(\delta_1, \dots, \delta_n),$$

$$P(t) = \text{diag}(p_1(t), \dots, p_n(t)).$$

Let q be a real number, $q > 0$. Define $\varphi^q: \mathbb{R} \longrightarrow \mathbb{R}$ by $\varphi^q(0) = 0$

and

$$\varphi^q(\tau) = (\operatorname{sgn} \tau) |\tau|^q.$$

for $\tau \neq 0$. (In particular, if q is a positive odd integer or its inverse, then $\varphi^q(\tau)$ is equal to τ^q .) For an element $x \in \mathbb{R}^n$ put

$$\Phi^q(x) = (\varphi^q(x_1), \dots, \varphi^q(x_n)) \in \mathbb{R}^n.$$

Proposition 5.1.3. Let s be a real number and let A be a T -periodic hyperbolic $n \times n$ matrix-valued function with the Morse index $(m, 1)$. Consider the equation

$$(5.2) \quad \dot{x} = A(t)x + \frac{1}{\|x\|^s} \Delta P(t) \Phi^q(x),$$

where $\|\cdot\|$ denotes the standard euclidean norm in \mathbb{R}^n . By r

($0 \leq r \leq n$) denote the number of positive elements among

$\delta_1, \dots, \delta_n$. Assume that either

$$q - s > 0, \quad q - s \neq 1$$

and r is arbitrary, or else

$$q - s < 0, \quad r = 0 \text{ or } n.$$

(i) If $m-1-r$ is odd then (5.2) has at least two distinct nonzero T -periodic solutions.

(ii) If $m-r$ is odd then (5.2) has at least two distinct nonzero $2T$ -periodic solutions.

Moreover, if either $q-s > 1$ and r is arbitrary, or else $q-s < 1$ and $r = 0$ or n , then the required periodic solutions are contained in $\mathbb{R}^n \setminus 0$.

Proof: Observe at the beginning that the right-hand side of (5.2) is odd with respect to x , hence the number of nonzero

periodic solutions is even or infinite. In particular, in order to prove (i) or (ii) it suffices to prove the existence of one such solution. Observe also, that if $q-s > 1$ then (5.2) satisfies the uniqueness of the Cauchy problem, hence the constant function 0 is the only solution which is not contained in $\mathbb{R}^n \setminus 0$.

Let W denote the strong T -periodic block for A considered in Proposition 3.3.1. At first let us assume that $q-s > 0$. In that case the right-hand side of (5.2) is defined in the whole \mathbb{R}^n , its value at the origin is equal to 0 for any t . Put

$$C = \{x \in \mathbb{R}^n : \forall i = 1, \dots, n: |x_i| \leq 1\},$$

$$D = \{x \in C : \exists j = 1, \dots, n: \delta_j > 0, |x_j| = 1\}.$$

By the assumptions, the set $\mathbb{R} \times C$ is a T -periodic block for

$$\dot{x} = \frac{1}{\|x\|^s} \Delta P(t) \Phi^q(x)$$

and $\mathbb{R} \times D$ is its exit set.

Consider the case $q-s > 1$. Proposition 3.4.1(i) guarantees the existence of a $\eta > 0$ such that $\mathbb{R} \times \eta C$ is a T -periodic block for (5.2) and

$$(\mathbb{R} \times \eta C)^- = \mathbb{R} \times \eta D.$$

By Proposition 3.4.1(iii) there exists an $\varepsilon > 0$ such that W_ε is a T -periodic block for (5.2). Moreover, we can assume that

$$W_\varepsilon \subseteq \mathbb{R} \times \eta C.$$

Since

$$\text{Lef}_{kT}(\mathbb{R} \times C, \mathbb{R} \times D) = (-1)^T,$$

for each k , by Remarks 3.3.2 and 3.4.2, and Theorem 4.2.2 we conclude the assertions (i) and (ii) for $q-s > 1$.

A similar argument applies in the case $0 < q-s < 1$. Indeed,

in that case the role of ε and η is reversed, now W_ε should be large and $\mathbb{R} \times \eta C$ should be small enough. This can be achieved by Proposition 3.4.1(ii),(iv). Since the Lefschetz numbers remain unchanged, the previous argument applies. The proof is finished in the case $q-s > 0$, $q-s \neq 1$.

Now we consider the reminded case $q-s \leq 0$. Observe, that in that case the previous argument fails. Since the right-hand side of (5.2) is defined in $\mathbb{R}^n \setminus 0$ only, W_ε is no longer a periodic block for any value of ε . However, we are still able to construct a periodic block, by removing from W_ε a sufficiently thick open tube surrounding the axis $\mathbb{R} \times \{0\}$. Below we present the details of that construction. The same argument can be also applied in the case $0 < q-s < 1$ under the assumption $r = 0$ or n , hence it yields the existence of a periodic solution in $\mathbb{R}^n \setminus 0$.

Let us follow the notation used in Section 3.3; in particular we consider the functions L^1, \dots, L^4 introduced there. Thus

$$W_\varepsilon = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \Lambda_\varepsilon^i(t, x) \leq 0, i = 1, \dots, 4\},$$

where $\Lambda_\varepsilon^i(t, x) = L^i(\varepsilon^{-1} \Omega(t)^{-1} x)$. By Proposition 3.4.1(ii), choose a large ε such that if $\Lambda_\varepsilon^i(t, x) = 0$ and $(t, x) \in W_\varepsilon$ then

$$\text{grad } \Lambda_\varepsilon^i(t, x) \cdot (1, A(t)x + \frac{1}{\|x\|^s} \Delta P(t) \Phi^q(x)) > 0$$

provided $i = 1, 2$, and

$$\text{grad } \Lambda_\varepsilon^i(t, x) \cdot (1, A(t)x + \frac{1}{\|x\|^s} \Delta P(t) \Phi^q(x)) < 0$$

provided $i = 3, 4$. Moreover, let ε be so large, that if $\|x\| \leq 2$ then $(t, x) \in \text{int } W_\varepsilon$ for every $t \in \mathbb{R}$.

Assume, that $r = 0$. For $\eta > 0$ define a function

$\Xi_\eta: \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\Xi_\eta(t, x) = 1 - \eta^{-1} \|x\|.$$

Similarly as above, by Proposition 3.4.1(iv), choose a small η ,

$0 < \eta < \frac{1}{2}$ such that

$$\text{grad } \Xi_\eta(t, x) \cdot (1, A(t)x + \frac{1}{\|x\|^s} \Delta P(t) \Phi^q(x)) > 0$$

Define

$$U = \{(t, x) \in \mathbb{R} \times (\mathbb{R}^n \setminus 0) : \Lambda_\varepsilon^i(t, x) \leq 0, i = 1, \dots, 4, \Xi_\eta(t, x) \leq 0\},$$

$$V = \{(t, x) \in Y : \Lambda_\varepsilon^i(t, x) = 0, i = 1 \text{ or } 2, \text{ or } \Xi_\eta(t, x) = 0\}.$$

In order to prove that U is a strong T -periodic block for (5.2)

and V is its exit set, one should find a time-dependent

vector-field generating a T -periodic process such that both U and

V consist of its trajectories.

Let $\alpha: \mathbb{R} \longrightarrow [0, 1]$ be a smooth function such that $\alpha(t) = 0$ if $t \leq \frac{1}{2}$ and $\alpha(t) = 1$ if $t \geq 2$. Recall the matrix-valued

T -periodic function H introduced in Section 3.3. Define

$h: \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$h(t, x) = \alpha(\|x\|) H(t)x.$$

This is the required vector-field. By ω denote the process

generated by h . The zero-sections of U and V are given by

$$U(0) = \{x \in \mathbb{R}^n \setminus 0 : L^i(\varepsilon^{-1}x) \leq 0, i = 1, \dots, 4, \|x\| \geq \eta\},$$

$$V(0) = \{x \in U(0) : L^i(\varepsilon^{-1}x) = 0, i = 1 \text{ or } 2, \text{ or } \|x\| = \eta\}.$$

Remark 3.2.3 imply that

$$\text{Lef}_T(U, U^-) = \text{Lef}(\omega_{(0, T)}|_{U(0)}) - \text{Lef}(\omega_{(0, T)}|_{V(0)}).$$

The set $U(0)$ is deformable to the unit sphere S^{n-1} by the radial projection. Using that deformation one can construct a homotopy

between the map $\omega_{(0, T)}|_{U(0)}$ and the identity on $U(0)$, hence

$$\text{Lef}(\omega_{(0,T)}|_{U(0)}) = \chi(S^{n-1}) = 1 - (-1)^n.$$

The map $\omega_{(0,T)}|_{V(0)}$ splits into two compounds; one of them is the identity on the η -sphere, the other one has the Lefschetz number equal to $\text{Lef}(\Omega(T)|_{\underline{B}})$ (in the notation of Section 3.3), hence

$$\text{Lef}(\omega_{(0,T)}|_{V(0)}) = \chi(S^{n-1}) + \text{Lef}(\Omega(T)|_{\underline{B}}) = 2 - (-1)^n - (-1)^{m-1}$$

by Remark 3.3.2. Thus

$$\text{Lef}_T(U, U^-) = (-1)^{m-1} - 1.$$

A similar argument leads to the conclusion, that

$$\text{Lef}_{2T}(U, U^-) = (-1)^m - 1.$$

In the case $r = n$ we argue in the same way, but here U^- is equal to V with the η -sphere removed. As a conclusion we obtain

$$\text{Lef}_T(U, U^-) = (-1)^{m-1} - (-1)^n,$$

$$\text{Lef}_{2T}(U, U^-) = (-1)^m - (-1)^n.$$

Theorem 4.2.1 guarantees the assertions (i) and (ii) in the case $q-s < 0$ and $r = 0$ or n . The proof is finished. ■

Example 5.1.4. By Proposition 5.4.3, each one of the equations

$$(\dot{x}, \dot{y}, \dot{z}) = (y - p_1(t)x^3, z - p_2(t)y^3, x - p_3(t)z^3)$$

$$(\dot{x}, \dot{y}, \dot{z}) = (y - p_1(t)x^{1/3}, z - p_2(t)y^{1/3}, x - p_3(t)z^{1/3})$$

$$(\dot{x}, \dot{y}, \dot{z}) = (y - p_1(t)\frac{x}{x^2+y^2+z^2}, z - p_2(t)\frac{y}{x^2+y^2+z^2}, x - p_3(t)\frac{z}{x^2+y^2+z^2})$$

has two distinct T -periodic solutions in $\mathbb{R}^3 \setminus 0$. Indeed, the Morse index of their linear parts is equal to $(1, 0)$.

5.2. A formula connecting the fixed point and Conley indices.

Corollary 2.4.1 has an application in the theory of isolated invariant sets (compare remarks in Section 2.2). It establishes a relation between the Conley index (which is the main tool in that theory) and the fixed point index. The Conley index of an isolated invariant set S of a local flow π (denoted by $h(\pi, S)$) is defined as the homotopy type of the pointed-space $(B/B^-, *)$, where $*$ denotes the point to which B^- is collapsed, and B is some isolating block containing S as its maximal invariant subset (in that case B is called a block isolating S). $h(\pi, S)$ is independent of the choice of B (compare [C2, p.50] or [Sm, p.475]).

Proposition 5.2.1. If π is a local flow on an ENR-space X , S is an isolated invariant set for π and there is a block B isolating S such that B and B^- are ENR's, then for every $T > 0$

$$\text{ind}(\pi_T, S_T) = \chi(h(\pi, S)),$$

where S_T denotes the set of T -periodic points contained in S .

Proof. By [R, p.57], if B is a block isolating S then

$$H(h(\pi, S)) = H(B, B^-)$$

for any homology functor H , hence

$$\chi(h(\pi, S)) = \chi(B) - \chi(B^-).$$

The result follows by Corollary 2.4.1. ■

The above proposition is essentially the same result as [Sr1, Th. 4.4] and also [M1, Th. 4.1], [M2, Th. 2.1] and [M3, Cor. 3].

It is not clear, whether the ENR-structure of X implies the

existence of a block B which fulfills the assumptions of Corollary 5.2.1 (however, according to [M4], the number $\chi(h(\pi, S))$ is correctly defined provided X is a compact ENR). That holds true if X is a 2-dimensional manifold (see [Sr1, Th. 3.3]), and if X is a differential manifold and π is smooth (see [WY], also remarks in Section 3.2). In the latter case an equivalent version of Corollary 5.2.1 was stated (in terms of the Hopf index - see [Hi]) in [Mc], and it follows from the generalized Poincaré index formula in [P]. That formula is a rediscovered (and modified) result due to Marston Morse in 1929, as it was pointed out in [Go]. A topological version of the formula was recently presented in [F].

If a local flow π is generated by a C^1 vector-field f on \mathbb{R}^n , U is an open set such that the set

$$K = \{x \in U: \pi_T(x) = x\}$$

is compact then, by [CMZ, Cor.2],

$$(5.3) \quad \text{ind}(\pi_T, K) = (-1)^n \text{deg}(0, f, U)$$

(compare also [KZ, Lem.13.2] and [Sr1, Th.5.1]). Thus Proposition 5.2.1 and (5.3) imply that

$$(-1)^n \text{deg}(0, f, \text{int } B) = \chi(h(\pi, S))$$

for each block B isolating S (compare [Sr1, Th.4.4] in the general case or [R, p.162] in the case f is a gradient). For simple application of that result to non-autonomous equations, see [Sr2, Th.4] (it is essentially the same as [M2, Th.4.2] and [W2, Th.4.1]).

We present a corollary of Proposition 5.1.2. Under slightly

different notation, V.I. Arnold refers that corollary to a result of Schnirelman and Nikishin.

Corollary 5.2.2 (compare [A1, p.419]). If a local flow π on a 2-dimensional manifold is gradient-like (see [C2, Sec.I.6.1]) then

$$\text{ind}(\pi_T, \{x\}) \leq 1$$

for each isolated rest point of π and $T > 0$.

Proof. An isolated rest point x of the gradient-like local flow π is also an isolated invariant set (compare [C2, Sec.I.6.2]). By [Sr1, Th. 3.3] there exists a block B isolating $\{x\}$ such that B is a ENR and B^- is a 1-dimensional manifold with boundary. By the Churchill's exact sequence (see [C2, Sec.IV.5.1]),

$$1 = \chi(\{x\}) = \chi(B, B^-) + \chi(a^-) = \chi(h(\pi, \{x\})) + \chi(a^-),$$

where a^- is the subset of B^- of points with negative semi-trajectories contained in B , and the Euler characteristic of a^- is defined. By the structure of B^- , the set a^- have to be homeomorphic to the union of a finite collection of circles and compact subsets of the real line. It follows, that $\chi(a^-)$ is always nonnegative. The result follows by Proposition 5.2.1. ■

Chapter 6. Planar polynomial equations with periodic coefficients.

In this chapter we assume that the function f in the right-hand side of (1.1) is a polynomial with respect to x having T -periodic functions of t as coefficients. In that case we call

(1.1) a polynomial equation with periodic coefficients. Because of convenience of complex numbers notation, we restrict ourselves to the case $n = 2$. In that case the right-hand side of (1.1) has the form $\sum a_{m,n}(t)x^m\bar{x}^n$ for some complex-valued T -periodic functions $a_{m,n}$, $m, n \in \mathbb{N}$ and almost all $a_{m,n}$ are equal to zero. In particular, each finite Fourier-Taylor expansion of an arbitrary smooth function $\mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, which is T -periodic in the first variable, is of that form with

$$a_{1,m}(t) = \sum c_{k,1,m} e^{2\pi i k t / kT},$$

for some $c_{k,1,m} \in \mathbb{C}$ and $k \in \mathbb{N}$. Our purpose is to look for those values of $c_{k,1,m}$, for which the existence of (nonzero) T -periodic solutions of (1.1) is guaranteed.

In Section 6.1 we present an idea of a construction of periodic blocks for the polynomial equations of some special form. This idea is investigated in the other sections. In Section 6.2 we introduce hyperbolic and elliptic polynomials, classes of homogeneous polynomials which will be of interest. The main theorems for this chapter are presented (without proofs) in Section 6.3. In Section 6.4 we give several concrete examples of applications of the theorems. In Section 6.5 we introduce functions and sets necessary for constructing periodic blocks generated by hyperbolic and elliptic polynomials. Proofs of the theorems from Section 6.3 are presented in Section 6.6. In Section 6.7 we consider planar rational equations with periodic coefficients. By the methods introduced in the previous sections, we establish a theorem on that equations, which cannot be reduced

to a result on polynomial equations by the change of coordinate z into $1/z$.

Sections 6.1 - 6.6 are based on the paper [Sr4], while Section 6.7 is based on [KS].

6.1. Geometric motivation. Let q be an integer, $q \geq 2$. We

consider the planar equation

$$(6.1) \quad \dot{z} = \frac{1}{q+1} iz + \bar{z}^q e^{it}$$

where $z \in \mathbb{C}$ and $t \in \mathbb{R}$. As an easy calculation shows, it has the form

$$(3.6), \text{ where } H(t) \text{ is the constant matrix } \begin{bmatrix} 0 & -1/(q+1) \\ 1/(q+1) & 0 \end{bmatrix} \text{ and}$$

$g(t, z) = \bar{z}^q$, hence the local process φ generated by (6.1) is the

composition of two local flows. The origin is the unique

nontrivial isolated invariant set for the local flow generated by

\bar{z}^q . In the case $q = 2$,

$$g(x, y) = (x^2 - y^2, -2xy),$$

its phase portrait is drawn in [C2, p.19]. From the phase portrait

one can deduce the existence of a block isolating the origin,

which is a regular hexagon centered at 0, such that its exit set

consists of three disjoint segments, one of which intersects

perpendicularly the positive x -semi-axis. By a similar

observation, for each q one can find a block B isolating zero,

which is a regular $2(q+1)$ -gon, B^- consists of $q+1$ disjoint

segments and both the sets B and B^- are invariant with respect to

the rotation by the angle $2\pi/(q+1)$. By Remark 2.2.5 the set

$$W = \{(t, (x, y)) \in \mathbb{R} \times \mathbb{R}^2:$$

$$(x \cos \frac{t}{q+1} + y \sin \frac{t}{q+1}, y \cos \frac{t}{q+1} - x \sin \frac{t}{q+1}) \in B\}$$

is a 2π -periodic isolating block for φ (actually, it is a strong periodic block for (6.1)) and

$$W^- = \{(t, (x, y)) \in \mathbb{R} \times \mathbb{R}^2:$$

$$(x \cos \frac{t}{q+1} + y \sin \frac{t}{q+1}, y \cos \frac{t}{q+1} - x \sin \frac{t}{q+1}) \in B^-\}$$

is its exit set. In the case $q = 3$ the block W is drawn on Figure

2. Since B is contractible, $k = 1, \dots, p$ a $2k\pi$ -monodromy

homeomorphism of W^- has no fixed points, and a $2(q+1)\pi$ -monodromy

homeomorphism of W^- is equal to the identity on B^- , (2.4) implies

$$\text{Lef}_{2k\pi}(W, W^-) = \begin{cases} 1 & \text{if } k = 1, \dots, q \bmod (q+1) \\ -q & \text{if } k = 0 \bmod (q+1) \end{cases}$$

Actually, it follows by Proposition 3.4.1(i), that if B is chosen sufficiently large, then W also remains a periodic block for the equation

$$(6.2) \quad \dot{z} = \bar{z}^q e^{it} + p(t, z)$$

provided p is of lower order at infinity than q . Theorem 4.2.1

immediately imply the existence of a $2k\pi$ -periodic solution of

(6.2) under assumption that p is continuous and $2k\pi$ -periodic in t .

In the forthcoming sections we make the above ideas more precise as well as we develop them in various directions. We introduce a class of homogeneous polynomials, called hyperbolic polynomials, which behave in a similar way as the monomial \bar{z}^q . In particular, the origin is the unique compact invariant set of their local flows. We show, that isolating blocks surrounding the origin can be represented by intersections of discs (however, in some cases the existence of blocks of polygonal shape, like in the above example, can also be deduced). By the method described in

Remark 2.2.5 we generate periodic blocks for a class of equations generalizing (6.2). In the way described in the proof of Proposition 5.1.2 we will deduce the existence of nonzero periodic solutions. For some equations we construct periodic blocks in the way represented by the last part of the proof of Proposition 5.1.3. The required form of the equations is generated by elliptic polynomials, another class of homogeneous polynomials introduced in the next section.

6.2. Planar homogeneous polynomials. By a polynomial we mean a nonzero planar vector-field, whose components are polynomials of the coordinates x and y . Let P be an homogeneous polynomial P of degree $m \in \mathbb{N}$ (we treat 0 as an element of \mathbb{N}). In particular, $P(\lambda z) = \lambda^m P(z)$ for each $z \in \mathbb{R}^2$ and $\lambda > 0$, hence it is also homogeneous in the sense of Section 3.4. Using the complex numbers notation we write P as

$$(6.3) \quad P(z) = \sum_{p,q \in \mathbb{N}, p+q=m} a_{p,q} z^p \bar{z}^q$$

for some $a_{p,q} \in \mathbb{C}$. We have the following obvious result (see also [A2, 21.E]):

Lemma 6.2.1. Let $k \in \mathbb{N}$ and assume that

$$(6.4) \quad a_{p,q} = 0 \text{ unless } p - q = 1 \bmod k.$$

If $k \geq 1$ then P is symmetrical with respect to the rotation by the angle $\frac{2\pi}{k}$ around the origin. If $k = 0$ then P is symmetrical with respect to any rotation around the origin. ■

Recall, that we denote by dot \cdot the usual scalar product in \mathbb{R}^2 . Put

$$Z_P = \{t \in [0, 2\pi): e^{it} \cdot P(e^{it}) = 0\},$$

$$\Lambda_P(t) = ie^{-it} P(e^{it}) \cdot \frac{d}{dt}(e^{-it} P(e^{it})).$$

If all the coefficients $a_{p,q}$ in (6.3) are real, then $e^{it} \cdot P(e^{it})$ is equal to $\sum a_{p,q} \cos(p-q-1)t$ and

$$(6.5) \quad \Lambda_P(t) = \sum_{(p,q), (r,s)} (r-s-1) a_{p,q} a_{r,s} \cos(q-p+r-s)t.$$

Definition 6.2.2. P is called hyperbolic (elliptic) if $\Lambda_P(t) < 0$ (resp. $\Lambda_P(t) > 0$) for each $t \in Z_P$.

In particular, each constant map is an hyperbolic polynomial (of degree 0). Presentation of other examples is postponed to Remark 6.2.6. Unless otherwise stated, in the remainder of this section we assume that P is hyperbolic or elliptic. Since we assumed also that P is homogeneous, $P^{-1}(0) = 0$ or $P^{-1}(0)$ is empty, we can introduce the following notion:

Definition 6.2.3. By the index of P (denoted by $\text{Ind } P$) we mean the degree at 0 of P in the whole plane, i.e.

$$\text{Ind } P = \deg(0, P, \mathbb{R}^2).$$

In fact, $\text{Ind } P$ is equal to the index of the unit circle (with the positive orientation) relative to P (compare [Le, Appendix

II]), which justify our terminology. The following observation makes the geometric meaning of the notion of hyperbolic or elliptic polynomial more clear.

Lemma 6.2.4. Assume, that P is hyperbolic (elliptic) and $t \in Z_P$, hence $P(e^{it}) = \lambda e^{it}$ for some $\lambda \neq 0$. Then there exists an $\varepsilon_0 > 0$ such that

$$\varepsilon \lambda e^{i(t+\varepsilon)} \cdot P(e^{i(t+\varepsilon)}) > 0$$

(resp. < 0) for any $\varepsilon \neq 0$, $-\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$.

Proof. The vector $ie^{-it}P(e^{it})$ is perpendicular to $e^{-it}P(e^{it})$, hence

$$\Lambda_P(t) = \lim_{0 \neq \varepsilon \rightarrow 0} \frac{1}{\varepsilon} ie^{-it}P(e^{it}) \cdot e^{-i(t+\varepsilon)}P(e^{i(t+\varepsilon)}).$$

Moreover,

$$\begin{aligned} ie^{-it}P(e^{it}) \cdot e^{-i(t+\varepsilon)}P(e^{i(t+\varepsilon)}) &= \lambda \cdot e^{-i(t+\varepsilon)}P(e^{i(t+\varepsilon)}) \\ &= \frac{\lambda}{2} (e^{-i(t+\varepsilon)}P(e^{i(t+\varepsilon)}) + e^{i(t+\varepsilon)}\overline{P(e^{i(t+\varepsilon)})}) \\ &= \lambda e^{i(t+\varepsilon)} \cdot P(e^{i(t+\varepsilon)}). \end{aligned}$$

Since $\Lambda_P(t) > 0$ (resp. $\Lambda_P(t) < 0$), the result follows. ■

In particular, Z_P is always finite. By the Poincaré-Bendixson formula (see [Le, p.211]),

$$(6.6) \quad \text{Ind } P = 1 - \frac{1}{2} \text{ card } Z_P, \quad (P \text{ is hyperbolic}).$$

$$(6.7) \quad \text{Ind } P = 1 + \frac{1}{2} \text{ card } Z_P, \quad (P \text{ is elliptic}).$$

Lemma 6.2.5. If for some $k \in \mathbb{N}$, $k \geq 2$, P is symmetrical with respect to the rotation by the angle $\frac{2\pi}{k}$ around the origin, then

Ind $P \neq 0$.

Proof. By the assumption, $\text{card } Z_P$ is divisible by k , hence, by (6.6) and (6.7), the lemma is proved in the case $k \neq 2$. The remaining case follows from Lemma 6.2.4. ■

Let $\varphi = \{\varphi_t\}$ be the local flow on \mathbb{R}^2 generated by P . Assume that P is hyperbolic. Lemma 6.2.4 and the homogeneity of P imply, that if $t \in Z_P$ then $|\varphi_s(e^{it})| > 0$ for any $s \neq 0$ such that $|s|$ is sufficiently small. Thus each point of the unit circle is an egress point, or an ingress point, or a point of outward tangency of its trajectory with respect to the unit disc. Again by homogeneity, it follows that any disc centered at 0 is an isolating block for φ and all the sectors around the origin are hyperbolic (see [Le, p.208]). If P is elliptic, an analogous argument shows that P has only elliptic sectors and only the inward tangency is possible. Those observations explain our terminology.

We assert, that 0 is the only periodic point of φ . Indeed, if $\varphi_T(z) = z$ for some $z \neq 0$, then the origin, as the unique stationary point, must be surrounded by the closed curve $\gamma = \{\varphi_t(z) : t \in \mathbb{R}\}$. Let C_a and C_b be two circles centered at zero, having radius a and, respectively b , $a \leq b$, such that γ is contained between them, and the intersections $\gamma \cap C_a$ and $\gamma \cap C_b$ are nonempty. It follows by the homogeneity of P , that both the intersections are finite and $a \neq b$. In that case each point of

$\gamma \cap C_a$ is a point of outward tangency with respect to the disc inside of C_a , hence P have to be hyperbolic. Analogously, analyzing the set $\gamma \cap C_b$ we conclude, that P is simultaneously elliptic, which is impossible.

By the above assertion we conclude that the fixed point index $\text{ind}(\varphi_T, 0) \in \mathbb{Z}$ is correctly defined for any $T > 0$. By (5.3), we obtain therefore another characterization of the index of P :

$$\text{Ind } P = \text{ind}(\varphi_T, 0).$$

Remark 6.2.6. Let P be an arbitrary nonconstant polynomial given by (6.3) and assume that it has at most two terms nonzero, i.e.

$$P(z) = az^{\overline{p-q}} + bz^{\overline{r-s}},$$

where $p+q = r+s \geq 1$ and $p \neq r$. Assume moreover that $a, b \in \mathbb{R}$. In that case

$$Z_P = \{t \in [0, 2\pi): a \cos(p-q-1)t + b \cos(r-s-1)t = 0\}$$

and

$$(6.8) \quad \Lambda_P(t) = (p-q-1)a^2 + (p-q+r-s-2)ab \cos(q-p+r-s)t + (r-s-1)b^2$$

(see (6.5)). Let us consider three simple situations.

(i) $b = 0$. Then P is hyperbolic if $p-q \leq 1$, elliptic if $p-q \geq 1$, and $\text{Ind } P = p-q$.

(ii) $p-q-1 = -(r-s-1)$. In that case P is hyperbolic provided $(p-q-1)(a^2-b^2) < 0$, P is elliptic if $(p-q-1)(a^2-b^2) > 0$. If $p-q \geq 1$ then $\text{Ind } P = p-q$ provided $a^2 > b^2$ and $\text{Ind } P = 2+q-p$ provided $a^2 < b^2$.

(iii) $r-s = 1$. Then

$$Z_p = \{t \in [0, 2\pi): a \cos(p-q-1) = -b\},$$

so we have to assume that $a \neq 0$. If $t \in Z_p$ then

$$\Lambda_p(t) = (p-q-1)(a^2 - b^2).$$

Determination of the possible values of $\text{Ind } P$ in that case is left to the reader.

6.3. Theorems on polynomial equations. In this section we assume that P is given by (6.3), $\varphi, T \in \mathbb{R}$, $T > 0$ and $k \in \mathbb{N}$. Consider the differential equation

$$(6.9) \quad \dot{z} = \sum_{p+q=m} a_{p,q} e^{i\varphi(q+1-p)t} z^p \bar{z}^q + b(t, z),$$

where $b: \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is continuous and T -periodic in t .

Theorem 6.3.1. Assume that $m \geq 2$, (6.4) is satisfied, and

$$(6.10) \quad b(t, z)/|z|^m \longrightarrow 0 \text{ as } |z| \longrightarrow \infty, \text{ uniformly in } t.$$

Assume moreover that P is hyperbolic and

$$\varphi k T = 2\pi\mu$$

for some $\mu \in \mathbb{Z}$. If $\text{Ind } P \neq 0$ then (6.9) has a T -periodic solution.

Remark 6.3.2. The condition $\text{Ind } P \neq 0$ is satisfied if $k \geq 2$ (by Lemma 6.2.1 and Lemma 6.2.5) or if $k = 0$ (because in that case $P(z) = a_{p,q} z^p \bar{z}^q$ and $p-q = 1$, hence, by Remark 6.2.6(i), $\text{Ind } P = 1$).

A proof of the above theorem as well as the proof of the following one is postponed until Section 6.5.

Let Q be another polynomial homogeneous of degree n ,

$$(6.11) \quad Q(z) = \sum_{r+s=n} c_{r,s} z^r \bar{z}^s$$

for some $c_{r,s} \in \mathbb{C}$. Let $\psi \in \mathbb{R}$. Consider the equation

$$(6.12) \quad \dot{z} = i\psi z + \sum_{p+q=m} a_{p,q} e^{i\psi(q+1-p)t} z^p \bar{z}^q + \sum_{r+s=n} c_{r,s} e^{i\psi(s+1-r)t} z^r \bar{z}^s + b(t, z).$$

Theorem 6.3.3. Assume that $1 \leq m < n$ and $l \in \mathbb{N}$. Suppose that

(6.4) and

$$(6.13) \quad c_{r,s} = 0 \text{ unless } r - s = 1 \bmod l,$$

are satisfied. Let

$$\phi kT = 2\pi\mu, \quad \psi lT = 2\pi\nu$$

for some $\mu, \nu \in \mathbb{Z}$, and

$$(6.14) \quad b(t, z)/|z|^n \longrightarrow 0 \text{ as } |z| \longrightarrow \infty, \text{ uniformly in } t,$$

$$(6.15) \quad b(t, z)/|z|^m \longrightarrow 0 \text{ as } 0 \neq |z| \longrightarrow 0, \text{ uniformly in } t.$$

Assume moreover that P is elliptic or hyperbolic and Q is hyperbolic. Then (6.12) has a nonzero T -periodic solution provided one of the following conditions holds:

- (a) $\text{Ind } P \neq 1$, $\mu = 0 \bmod k$, and $\nu \neq 0 \bmod l$,
- (b) $\text{Ind } Q \neq 1$, $\mu \neq 0 \bmod k$, and $\nu = 0 \bmod l$,
- (c) $\text{Ind } P \neq \text{Ind } Q$, $\mu = 0 \bmod k$, and $\nu = 0 \bmod l$,

As we mentioned above, a proof of Theorem 6.3.3 is presented in Section 6.5. We conclude the current section by two corollaries from the above theorems.

Consider the equation

$$(6.16) \quad \dot{z} = \sum_{0 \leq k+l < r+s} b_{k,l}(t) z^k \bar{z}^l + ce^{i\omega t} z^{r-s}$$

in which $k, l, r, s \in \mathbb{N}$, $c, \omega \in \mathbb{R}$, $c \neq 0$ and $b_{k,l}$ is a continuous, T -periodic complex-valued function.

Corollary 6.3.4. Assume that $r+s \geq 2$ and one of the following two conditions is satisfied:

- (a) $\omega = 0$, $r-s \leq 1$, $r \neq s$ and T is arbitrary,
- (b) $\omega \neq 0$, $r-s \leq -1$ and T is a multiplicity of $\frac{2\pi}{\omega}$.

Then (6.16) has a T -periodic solution.

Proof: Put $P(z) = cz^{r-s}$. Since $r-s \leq 1$, P is hyperbolic and $\text{Ind } P = r-s$ (compare Remark 6.2.6(i)). Since it is assumed that $r-s \neq 0$, $\text{Ind } P$ is nonzero. Assume the condition (a). Then (6.16) is of the form (6.9) with $\varphi = 0$, hence Theorem 6.3.1 (with $k = 1$) imply the result. In the case (b) we put $k = s+1-r \geq 2$ and $\varphi = \omega/k$, hence the result follows again by Theorem 6.3.1. ■

Consider a special case of (6.16):

$$(6.17) \quad \dot{z} = i\varphi z + ae^{i\varphi(q+1-p)t} z^{p-q} + \sum_{p+q < k+l < r+s} b_{k,l}(t) z^k \bar{z}^l + ce^{i\omega t} z^{r-s}$$

(where $p, q \in \mathbb{N}$, $1 \leq p+q < r+s$, $a, \varphi \in \mathbb{R}$, $a \neq 0$).

Corollary 6.3.5. The equation (6.17) has a nonzero T -periodic solution provided one of the following conditions is satisfied:

(a) $\varphi(q+1-p) = 0$, $\omega = 0$, $r-s \leq 1$, $p-q \neq r-s$, and T is arbitrary,

(b) $\varphi(q+1-p) \neq 0$, $\omega = 0$, and

(ba) $r-s = 1$ and $T = \frac{2\pi\mu}{\varphi}$ for some $\mu \in \mathbb{Z}$,

(bb) $p-q = r-s \leq 0$ and $T = \frac{2\pi\mu}{\varphi(q+1-p)}$ for some $\mu \in \mathbb{Z}$,

$\mu \neq 0 \pmod{q+1-p}$,

(bc) $p-q \neq r-s$, $r-s \leq 0$, and $T = \frac{2\pi\mu}{\varphi(q+1-p)}$ for some $\mu \in \mathbb{Z}$,

(c) $\varphi(q+1-p) = 0$, $\omega \neq 0$, $r-s \leq 0$, and

(ca) $p-q = 1$ and $T = \frac{2\pi(s+1-r)\nu}{\omega}$ for some $\nu \in \mathbb{Z}$,

(cb) $p-q = r-s$ and $T = \frac{2\pi\nu}{\omega}$ for some $\nu \in \mathbb{Z}$, $\nu \neq 0 \pmod{s+1-r}$,

(cc) $p-q \neq r-s$, $p-q \neq 1$, and $T = \frac{2\pi\nu}{\omega}$ for some $\nu \in \mathbb{Z}$,

(d) $(q+1-p)\varphi\omega \neq 0$, $r-s \leq 0$, $T = \frac{2\pi\mu}{\varphi(q+1-p)} = \frac{2\pi\nu}{\omega}$ for some

$\mu, \nu \in \mathbb{Z}$, and

(da) $\mu = 0 \pmod{q+1-p}$ and $\nu \neq 0 \pmod{s+1-r}$,

(db) $\mu \neq 0 \pmod{q+1-p}$ and $\nu = 0 \pmod{s+1-r}$,

(dc) $p-q \neq r-s$, $\mu = 0 \pmod{q+1-p}$, and $\nu = 0 \pmod{s+1-r}$.

Proof: Put $P(z) = az^{\frac{p-q}{\varphi}}z^q$ and $Q(z) = cz^{\frac{r-s}{\omega}}z^s$. Like in the previous proof, by Remark 6.2.6(i), we match the assumptions to the corresponding assumptions of Theorem 6.3.3. The details are left to the reader. ■

6.4. Examples. We present examples illustrating the results stated in the previous section. Other examples can be easily obtained from the equations (6.16) and (6.17), they are based on the choice of constants indicated in Corollaries 6.3.4 and 6.3.5,

respectively.

Example 6.4.1. For $q \geq 2$ the equations

$$\dot{z} = \bar{z}^q + p(t),$$

$$\dot{z} = \bar{z}^q e^{it} + p(t)$$

have $2k\pi$ -periodic solutions, provided $k \in \mathbb{Z}$ and p is a $2k\pi$ -periodic continuous complex-valued function.

Indeed, the assertion is a consequence of Corollary 6.3.4.

Example 6.4.2. The equation

$$\dot{z} = 2e^{8it} \bar{z}^{-7} + e^{-2it} \bar{z}^{5-2} + 1$$

has a π -periodic solution.

Indeed, $P(z) = 2\bar{z}^7 + z^5 \bar{z}^{-2}$ is a hyperbolic polynomial by the formula (6.8) and $\text{Ind } P \neq 0$ by Remark 6.3.2 for $k = 2$. The result follows by Theorem 6.3.1 with $\varphi = 1$, $k = 2$, and $\mu = 1$.

Example 6.4.3. Assume that $q \geq 1$. Then

$$\dot{z} = z^q + p(t)z^{q+1} + \bar{z}^{q+2}$$

has a nonzero T -periodic solution provided p is a T -periodic continuous function.

The assertion follows by Corollary 6.3.5(a).

Example 6.4.4. Assume that $r \geq 2$. The equation

$$\dot{z} = \bar{z}^r e^{it} + z$$

has at least $r+1$ distinct nonzero $2(r+1)\pi$ -periodic solutions.

This is a consequence of Corollary 6.3.5(c)(ca) and the

symmetry of the right-hand side with respect to the rotation by the angle $2\pi/(r+1)$.

Example 6.4.5. Assume that $2 \leq q < r$. The equation

$$\dot{z} = \bar{z}^r e^{it} + z^q$$

has a nonzero 2π -periodic solution. Moreover, if $r + 1$ is a multiplicity of $q - 1$ and $q \geq 3$ (for example, $q = 3$ and $r = 4$), then there are at least $q - 1$ distinct such solutions.

This assertion follows by Corollary 6.3.5(c)(cc) and by Lemma 6.2.1.

Example 6.4.6. Assume that $1 \leq q < r$. The equation

$$\dot{z} = \bar{z}^r e^{it} + \bar{z}^q$$

has a nonzero 2π -periodic solution. Moreover, if $r + 1$ is a multiplicity of $q + 1$ (for example, $q = 1$ and $r = 3$), then there are at least $q + 1$ distinct such solutions.

This assertion is again a consequence of Corollary 6.3.5(c)(cc) and by Lemma 6.2.1.

Example 6.4.7. The equation

$$\dot{z} = 2e^{8it} \bar{z}^{-7} + e^{-2it} \bar{z}^{5-2} + |z|^q$$

has a nonzero π -periodic solution, provided q is a nonnegative even integer.

Indeed, by the argument in Example 6.3.2 and by Theorem 6.3.3(a) with $\varphi = 0$, $k = 1$, $\mu = 0$, $\psi = 1$, $l = 2$, and $\nu = 1$, it suffices to observe, that $|z|^q$ is a hyperbolic polynomial and its

index is equal to 0.

Example 6.4.8. The equation

$$\dot{z} = iz + 2e^{-3it}z^4 + e^{3it}\bar{z}^3 + z\bar{z}^5 + 2z^2\bar{z}^4$$

has a nonzero $2\pi/3$ -periodic solution.

Indeed, put $P(z) = 2z^4 + z\bar{z}^3$ and $Q(z) = z\bar{z}^5 + 2z^2\bar{z}^4$. By

Remark 6.2.6(ii), P is elliptic and Q is hyperbolic, $\text{Ind } Q = -2$.

The result follows by Theorem 6.3.3(b) with $\varphi = 1$, $k = 3$, $\mu = 1$, $\psi = 0$, $l = 1$, and $\nu = 0$.

Example 6.4.9. The equation

$$\dot{z} = 2iz + e^{6it}(\bar{z}^2 + \bar{z}^5)$$

has a nonzero π -periodic solution.

The conclusion follows by Corollary 6.3.5(d)(da) with $\mu = \nu = 3$.

Example 6.4.10. The equation

$$\dot{z} = -iz + e^{2it}(z^3 + \bar{z}^4)$$

has 2π -periodic and 5π -periodic nonzero solutions. (However, they can be distinct, hence we cannot predict the existence of a π -periodic solution.)

Indeed, in order to obtain the existence of the required solutions it suffices to apply Corollary 6.3.5(d)(da), (db) with $\mu = \nu = 2$ (or 5, respectively).

In the last two examples we proved the existence of

subharmonic periodic solutions. By the geometric method we cannot prove the existence of harmonic ones in the considered case. It is of interest to know whether such solutions actually exist.

6.5. Auxiliary sets and functions. In this section P is a fixed elliptic or hyperbolic polynomial, homogeneous of degree m . In order to construct strong periodic blocks generated by modifications of P , we introduce here several sets and functions associated with P .

Assume that u, v are real numbers such that $0 < v - u < 2\pi$, $P(e^{iu}) = \alpha e^{iu}$, $P(e^{iv}) = \beta e^{iv}$ (hence α and β are different from zero) and $P(e^{it}) \cdot e^{it} \neq 0$ for every $t \in (u, v)$. Let $\eta \neq 0$ be such that

$$(6.18) \quad \eta \cos \frac{u-v}{2} > -\frac{1}{2}.$$

Define a function $L_{P,\eta}^{u,v}: \mathbb{R}^2 \longrightarrow \mathbb{R}$ (shortly denoted by L_η) by

$$L_\eta(z) = (\operatorname{sgn} \eta) (|z + \eta e^{i(u+v)/2}|^2 - 1 - \eta^2 - 2\eta \cos \frac{u-v}{2}).$$

Define arcs $D_{P,\eta}^{u,v}$ and sectors $E_{P,\eta}^{u,v}$ (or D_η , E_η in abbreviated form) as

$$D_\eta = \{z \in \mathbb{C}: u \leq \arg z \leq v, L_\eta(z) = 0\},$$

$$E_\eta = \{z \in \mathbb{C}: u \leq \arg z \leq v, L_\eta(z) \leq 0\}.$$

where \arg is a branch of the argument. Obviously, both the points e^{iu} and e^{iv} are contained in D_η .

Lemma 6.5.1. If $\eta > 0$ then $L_\eta(0) < 0$ and

$$D_\eta = \{z \in \mathbb{C}: L_\eta(z) = 0, -\eta + (1 + \eta^2 + 2\eta \cos \frac{u-v}{2})^{1/2} \leq |z| \leq 1\},$$

$$E_\eta \subseteq \{z \in \mathbb{C}: |z| \leq 1\},$$

$$\{z \in \mathbb{C}: v < \arg z < u+2\pi, |z| \leq 1\} \subseteq \{L_\eta < 0\}.$$

If $\eta < 0$ then $L_\eta(0) > 0$ and

$$D_\eta = \{z \in \mathbb{C}: L_\eta(z) = 0, 1 \leq |z| \leq -\eta + (1 + \eta^2 + 2\eta \cos \frac{u-v}{2})^{1/2}\},$$

$$E_\eta \subseteq \{z \in \mathbb{C}: |z| \geq 1\},$$

$$\{z \in \mathbb{C}: v < \arg z < u+2\pi, |z| \geq 1\} \subseteq \{L_\eta < 0\}.$$

Proof. By (6.18) the required estimates on $L_\eta(0)$ are obvious.

Let $z = \lambda e^{it}$ for some $\lambda \geq 0$. Put $\tau = t - (u+v)/2$. Then

$$L_\eta(z) = (\operatorname{sgn} \eta) (\lambda^2 + 2\lambda\eta \cos \tau - 1 - 2\eta \cos \frac{u-v}{2}).$$

Again by (6.18), the unique nonnegative solution of $L_\eta(z) = 0$ is given by

$$\lambda = -\eta \cos \tau + (1 + \eta^2 \cos^2 \tau + 2\eta \cos \frac{u-v}{2})^{1/2}.$$

If $t \in [u, v]$, hence $\tau \in [-\frac{u-v}{2}, \frac{v-u}{2}] \subseteq (-\pi, \pi)$, then the required properties of D_η and E_η follow. The remaining inclusions also follow by the above formulas. ■

Lemma 6.5.2. The product $\alpha\beta$ is negative and there exists an $\eta_P^{u,v} > 0$ such that for any $z \in D_\eta$

$$\operatorname{grad} L_\eta(z) \cdot P(z) > 0 \text{ if } \alpha > 0 \text{ (or, equivalently: } \beta < 0),$$

$$\operatorname{grad} L_\eta(z) \cdot P(z) < 0 \text{ if } \alpha < 0 \text{ (or, equivalently: } \beta > 0),$$

provided $0 < \eta \leq \eta_P^{u,v}$ in the case P is hyperbolic and

$$-\eta_P^{u,v} \leq \eta < 0 \text{ if } P \text{ is elliptic.}$$

Proof. Assume that P is hyperbolic and $\alpha > 0$. Let $z \in D_\eta$, hence $z = \lambda e^{it}$ for some $\lambda > 0$ and $t \in [u, v]$.

$$\operatorname{grad} L_\eta(z) \cdot P(z) = (\lambda e^{it} + \eta e^{i(u+v)/2}) \cdot P(\lambda e^{it})$$

$$= \lambda^m (\lambda e^{it} \cdot P(e^{it}) + \eta e^{i(u+v)/2} \cdot P(e^{it})).$$

By Lemma 6.2.4, there exists an η_0 such that if $0 < \eta \leq \eta_0$ then

$$e^{i(u+\eta)} \cdot P(e^{i(u+\eta)}) > 0,$$

hence, by assumptions on u and v ,

$$e^{it} \cdot P(e^{it}) > 0 \quad (\forall t \in (u, v)).$$

Thus, again by Lemma 6.2.4, we obtain that $\beta < 0$, hence the first assertion follows. We have also

$$e^{i(u+v)/2} \cdot P(e^{iu}) > 0, \quad e^{i(u+v)/2} \cdot P(e^{iv}) > 0.$$

Indeed,

$$e^{i(u+v)/2} \cdot \alpha e^{iu} = \alpha \operatorname{Re} i e^{i(u-v)/2}.$$

Since $0 < (u-v)/2 < \pi$, the first inequality follows. Similarly we prove the second one. Now it is easy to verify, that if η is positive and sufficiently small then

$$\lambda e^{it} \cdot P(e^{it}) + \eta e^{i(u+v)/2} \cdot P(e^{it}) > 0$$

for any $t \in [u, v]$. This finishes the proof in the considered case.

In the other cases the proof is analogous. ■

The above lemma will allow us to obtain inequalities of the form (3.10), (3.11) for hyperbolic or elliptic polynomials.

Let Z_p be nonempty and let it consist of N elements. Denote the elements of Z_p by u^1, \dots, u^N and assume that

$$0 \leq u^1 < u^2 < \dots < u^N < 2\pi.$$

For each $j = 1, \dots, N$ and $l \in \mathbb{Z}$ put also

$$u^{j+lN} = u^j + 2\pi l,$$

hence u^j is defined for any integer j . It follows that

$$(6.19) \quad L_{P, \eta}^{u^j, u^{j+1}} = L_{P, \eta}^{u^k, u^{k+1}} \quad \text{if } j = k \bmod N,$$

and analogous relations hold for the other notions introduced at the beginning of this section.

Let j be an integer. Then $P(e^{iu^j}) = \alpha_j e^{iu^j}$ for some real number α_j . Obviously $\alpha_j = \alpha_{j+N}$. By Lemma 6.5.2,

$$(6.20) \quad \alpha_j \alpha_{j+1} < 0 \text{ for each } j \in \mathbb{Z},$$

which implies in particular, that N is even (compare also [Le, p.212]). Choose a real number δ such that:

(C1) If P is hyperbolic then $\delta > 0$ and

$$\delta \leq \min \{ \eta_{P, \delta}^{u^1, u^2}, \eta_{P, \delta}^{u^2, u^3}, \dots, \eta_{P, \delta}^{u^N, u^{N+1}} \},$$

$$\frac{1}{2} < -\delta + (1 + \delta^2 + 2\delta \cos \frac{u^j + u^{j+1}}{2}) \text{ for } j = 1, \dots, N.$$

(C2) If P is elliptic then $\delta < 0$ and

$$\delta \geq - \min \{ \eta_{P, \delta}^{u^1, u^2}, \eta_{P, \delta}^{u^2, u^3}, \dots, \eta_{P, \delta}^{u^N, u^{N+1}} \},$$

$$-\delta + (1 + \delta^2 + 2\delta \cos \frac{u^j + u^{j+1}}{2}) < 2 \text{ for } j = 1, \dots, N.$$

(Here $\eta_{P, \delta}^{u^j, u^{j+1}}$ satisfies Lemma 6.5.2.)

Let $z \in \mathbb{R}^2$. If $\alpha_1 > 0$, define (using (6.19))

$$L_P^-(z) = \max \{ L_{P, \delta}^{u^j, u^{j+1}}(z), j \text{ is odd} \},$$

$$L_P^+(z) = \max \{ L_{P, \delta}^{u^j, u^{j+1}}(z), j \text{ is even} \}.$$

If $\alpha_1 < 0$, we reverse the definitions of $L_P^\pm(z)$, i.e.

$$L_P^-(z) = \max \{ \dots, L_{P, \delta}^{u^2, u^3}(z), \dots \}, L_P^+(z) = \max \{ \dots, L_{P, \delta}^{u^1, u^2}(z), \dots \}.$$

Assume now that Z_P is empty. In that case P is simultaneously hyperbolic and elliptic. As above, we associate to P two functions L_P^- and L_P^+ . In order to simplify formulations of some results presented below, we assume the following convention on the functions, which recognizes whether P is treated as a

hyperbolic or elliptic polynomial.

At first let us consider P as a hyperbolic polynomial. In the case $P(1) \cdot 1 > 0$ we define

$$L_P^-(z) = |z|^2 - 1, \quad L_P^+(z) = -1.$$

In the opposite case $P(1) \cdot 1 < 0$ put

$$L_P^-(z) = -1, \quad L_P^+(z) = |z|^2 - 1.$$

Let P be regarded now as an elliptic polynomial. If $P(1) \cdot 1 > 0$ put

$$L_P^-(z) = -1, \quad L_P^+(z) = 1 - |z|^2$$

and if $P(1) \cdot 1 < 0$ put

$$L_P^-(z) = 1 - |z|^2, \quad L_P^+(z) = -1.$$

Thus the functions L_P^\pm are defined in every case. Define also the sets C_P , D_P^+ and D_P^- by

$$C_P = \{z \in \mathbb{R}^2: L_P^+(z) \leq 0, L_P^-(z) \leq 0\}$$

$$D_P^\pm = \{z \in C_P: L_P^\pm(z) = 0\}$$

Consequences of the choice of the above functions and sets are summarized in the following result:

Lemma 6.5.3. The functions L_P^\pm are smooth in some neighborhood of D_P^\pm , for each $z \in D_P^+ \cap D_P^-$ the set $\{\text{grad } L_P^+(z), \text{grad } L_P^-(z)\}$ is linearly independent and the following inequalities are satisfied:

$$(6.21) \quad \text{grad } L_P^-(z) \cdot P(z) > 0 \quad (\forall z \in D_P^-),$$

$$(6.22) \quad \text{grad } L_P^+(z) \cdot P(z) < 0 \quad (\forall z \in D_P^+).$$

Moreover, $\partial C_P = D_P^+ \cup D_P^-$ and if P is hyperbolic, then C_P is compact, contained in the closed unit disc, $0 \in \text{int } C_P$ and

$$\partial C_P \subseteq \{z \in \mathbb{C}: \frac{1}{2} \leq |z| \leq 1\}.$$

If P is elliptic then $\text{int } C_P$ is compact, contained in the unit

disc, $0 \in \setminus C_P$ and

$$\partial C_P \subseteq \{z \in \mathbb{C}: 1 \leq |z| \leq 2\}.$$

Proof: Only the case Z_P is nonempty requires a proof. We apply the above notation concerning that case. In particular,

$$C_P = \{z \in \mathbb{R}^2: \max \{L_{P,\delta}^{u^1, u^2}(z), L_{P,\delta}^{u^2, u^3}(z), \dots, L_{P,\delta}^{u^N, u^{N+1}}(z)\} \leq 0\}.$$

Assume that P is hyperbolic. By Lemma 6.5.1, $0 \in \text{int } C_P$ in that case. We prove that C_P is contained in the unit disc. Let $z \in C_P$, hence $u^j \leq \arg z \leq u^{j+1}$ for some $j = 1, \dots, N$. By the above description of C_P we conclude, that $z \in E_{P,\eta}^{u^j, u^{j+1}}$, hence, by Lemma 6.5.1, $|z| \leq 1$. The same proof is valid in the other cases, so we have the required property of C_P . Assume moreover, that $\alpha_1 > 0$ (recall, that α_1 is given by the relation $P(e^{iu^1}) = \alpha_1 e^{iu^1}$). By Lemma 6.5.2, (6.20) and (C1),

$$(6.23) \quad \text{grad } L_{P,\delta}^{u^j, u^{j+1}}(z) \cdot P(z) > 0 \quad (\forall z \in D_{P,\delta}^{u^j, u^{j+1}}, j \text{ odd}),$$

$$(6.24) \quad \text{grad } L_{P,\delta}^{u^j, u^{j+1}}(z) \cdot P(z) < 0 \quad (\forall z \in D_{P,\delta}^{u^j, u^{j+1}}, j \text{ even}).$$

We assert, that

$$(6.25) \quad L_P^- = L_{P,\delta}^{u^j, u^{j+1}} \quad (\text{in some neighborhood of } D_{P,\delta}^{u^j, u^{j+1}}, j \text{ odd}).$$

We prove (6.25) for $j = 1$. In the other cases the proof is

analogous. If $z \in D_{P,\delta}^{u^1, u^2}$ then $L_{P,\delta}^{u^1, u^2}(z) = 0$. Since, by the definition and by Lemma 6.5.1,

$$D_{P,\delta}^{u^1, u^2} \subseteq \{z \in \mathbb{C}: u^4 < \arg z < u^3 + 2\pi\} \cap \{|z| \leq 1\},$$

again Lemma 6.5.1 implies that the set $D_{P,\delta}^{u^1, u^2}$ is contained in

$\{L_{P,\delta}^{u^3,u^4} < 0\}$, hence

$$0 = L_{P,\delta}^{u^1,u^2}(z) > L_{P,\delta}^{u^3,u^4}(z)$$

for each $z \in D_{P,\delta}^{u^1,u^2}$. By applying of that argument to

$L_{P,\delta}^{u^5,u^6}, \dots, L_{P,\delta}^{u^{N-1},u^N}$, we obtain (6.25) in the considered case.

Similar argument applied to $L_{P,\delta}^{u^2,u^3}, \dots, L_{P,\delta}^{u^N,u^1}$ shows that for each $z \in D_{P,\delta}^{u^1,u^2}$,

$$L_P^+(z) \leq L_{P,\delta}^{u^1,u^2}(z) = 0.$$

Thus

$$(6.26) \quad D_{P,\delta}^{u^1,u^2} \subseteq \{L_P^- = 0\} \cap \{L_P^+ \leq 0\} = D_P^-.$$

By (6.23) and (6.25), in order to prove (6.21) it suffices to show that

$$(6.27) \quad D_P^- = D_{P,\delta}^{u^1,u^2} \cup \dots \cup D_{P,\delta}^{u^{N-1},u^N}.$$

We have just shown (6.26), an analogous proof gives also

$D_{P,\delta}^{u^j,u^{j+1}} \subseteq D_P^-$ for any other odd number j , hence the right side of

(6.27) is contained in the left one. Let $z \in D_P^-$. Without loss of generality we can assume that $L_{P,\delta}^{u^1,u^2}(z) = 0$. Suppose that

$u^2 < \arg z < u^1 + 2\pi$. Then, by Lemma 6.5.1, $|z| > 1$. Since we have

already proved that C_P (and thus D_P^-) is contained in the unit

disc, we conclude that $z \in D_{P,\delta}^{u^1,u^2}$, hence (6.27) follows. Using

(6.24) one proves the inequality (6.22) by the same argument. It

follows by (6.21) that $\text{grad } L_P^-$ is nonzero in D_P^- , and by (6.22)

that $\text{grad } L_P^+$ is nonzero in D_P^+ , hence the boundary of C_P is the

union of D_P^+ and D_P^- . By Lemma 6.5.1, (C1), and (6.27), the boundary

has to be contained in the annulus $\{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 1\}$. Finally,

let us prove that $\text{grad } L_P^+(z)$ and $\text{grad } L_P^-(z)$ are linearly independent at a point z in the intersection of D_P^+ and D_P^- . Indeed, by (6.27) and the similar formula on D_P^+ , we conclude that

$$D_P^+ \cap D_P^- = \{e^{iu^1}, \dots, e^{iu^N}\}.$$

By (6.25) and its counterpart for an even number j , one can compute both the gradients at each point e^{iu^j} and check their linear independence. Thus the proof of the lemma is finished in the case P hyperbolic and $\alpha_1 > 0$. The proof in the other cases is analogous. ■

For $\varepsilon, \varphi \in \mathbb{R}$, $\varepsilon > 0$, define new functions $\Lambda_{P,\varphi,\varepsilon}^+$, $\Lambda_{P,\varphi,\varepsilon}^-$ and sets $W_{P,\varphi,\varepsilon}$, $Y_{P,\varphi,\varepsilon}^+$ and $Y_{P,\varphi,\varepsilon}^-$ by

$$\Lambda_{P,\varphi,\varepsilon}^\pm(t, z) = L_P^\pm(\varepsilon^{-1} e^{-i\varphi t} z),$$

$$W_{P,\varphi,\varepsilon} = \{(t, z) \in \mathbb{R} \times \mathbb{R}^2 : \Lambda_{P,\varphi,\varepsilon}^+(t, z) \leq 0, \Lambda_{P,\varphi,\varepsilon}^-(t, z) \leq 0\}$$

$$= \{(t, e^{i\varphi t} w) \in \mathbb{R} \times \mathbb{R}^2 : w \in \varepsilon C_P\},$$

$$Y_{P,\varphi,\varepsilon}^\pm = \{(t, z) \in W_{P,\varphi,\varepsilon} : \Lambda_{P,\varphi,\varepsilon}^\pm(t, z) = 0\}$$

$$= \{(t, e^{i\varphi t} w) \in \mathbb{R} \times \mathbb{R}^2 : w \in \varepsilon D_P^\pm\}.$$

Through reminder of this section we assume that P is given by (6.3), $\varphi \in \mathbb{R}$, $k \in \mathbb{N}$, and (6.4) is satisfied, i.e.

$$a_{p,q} = 0 \text{ unless } p-q = 1 \bmod k.$$

The following result explains our choice of the functions L_P^\pm .

Lemma 6.5.4. Let $z \in \mathbb{R}^2$. If $k \geq 1$, $\varepsilon > 0$ and $\varphi \neq 0$ then $\Lambda_{P,\varphi,\varepsilon}^\pm(t, z)$ is $\frac{2\pi}{|\varphi|k}$ -periodic in t . If $\varphi k = 0$ then $\Lambda_{P,\varphi,\varepsilon}^\pm(t, z)$ is independent of t .

Proof: The result is obvious if $k = 1$, $\varphi = 0$ or Z_P is empty. The latter possibility holds if $k = 0$, since in that case $P(z) = a|z|^{2q}z$ for some $a \in i\mathbb{R}$ and $q = (m-1)/2$. Thus we can assume that $\varphi \neq 0$, $k \geq 2$ and $Z_P \neq \emptyset$. Since P is symmetrical with respect to the rotation by the angle $\frac{2\pi}{k}$ (compare Lemma 6.2.1), $N = kM$ for some $M \in \mathbb{N}$. The number M is even. Indeed, by the symmetry of P , $\alpha_j = \alpha_{j+M}$, hence (6.20) implies the assertion. It follows also that

$$(6.28) \quad u^{j+1} - u^j = u^{j+1+\mu M} - u^{j+\mu M},$$

$$(6.29) \quad e^{i(u^j + 2\pi\mu/k)} = e^{iu^{j+\mu M}}$$

for any $j, \mu \in \mathbb{Z}$. Since

$$1 + \delta^2 + 2\delta \cos \frac{u^{1+1} - u^1}{2} = |e^{iu^1} + \delta e^{i(u^1 + u^{1+1})/2}|^2$$

for each $l \in \mathbb{Z}$, by (6.28), (6.29) we have the equation

$$L_{P,\delta}^{u^j, u^{j+1}}(e^{-i\varphi(t+2\pi/|\varphi|k)}z) = L_{P,\delta}^{u^{j+(\text{sgn } \varphi)M}, u^{j+(\text{sgn } \varphi)M+1}}(e^{-i\varphi}z).$$

Since M is even, the above equation and the definition of L_P^\pm imply

$$L_P^\pm(e^{-i\varphi(t+2\pi/|\varphi|k)}z) = L_P^\pm(e^{-i\varphi t}z),$$

hence the result is proved. ■

Now we present some periodicity properties of the sets defined above. In the sequel we assume that $\varepsilon > 0$.

Lemma 6.5.5. Let P be a hyperbolic or elliptic polynomial.

The set $Y_{P,\varphi,\varepsilon}^\pm$ is a T -periodic proper ENR provided the condition

$$(6.30) \quad \exists \mu \in \mathbb{Z}: \varphi k T = 2\pi\mu$$

is satisfied. Moreover,

$$\text{Lef}_T(Y_{P,\varphi,\varepsilon}^\pm) = \begin{cases} 0, & \text{if } \mu \neq 0 \bmod k \\ 1 - \text{Ind } P, & \text{if } P \text{ is hyperbolic, } \mu = 0 \bmod k \\ \text{Ind } P - 1, & \text{if } P \text{ is elliptic, } \mu = 0 \bmod k \end{cases}$$

Proof. The set $Y_{P,\varphi,\varepsilon}^\pm$ consists of trajectories of

$$\dot{z} = \varphi iz,$$

hence the required periodicity follows by Lemma 6.5.4. The zero-section εD_P^\pm of $Y_{P,\varphi,\varepsilon}^\pm$ is an empty set or a circle, provided Z_P is empty, and it is a collection of $\frac{1}{2}$ card Z_P disjoint arcs in the opposite case (compare (6.27)). In any case it is a compact ENR. If $\varphi k = 0$ then

$$Y_{P,\varphi,\varepsilon}^\pm = \mathbb{R} \times \varepsilon D_P^\pm,$$

hence

$$\text{Lef}_T(Y_{P,\varphi,\varepsilon}^\pm) = \chi(D_P^\pm) = \frac{1}{2} \text{card } Z_P.$$

The same holds true if $\varphi k \neq 0$ and $\mu = 0 \bmod k$. Indeed, if $\varphi k \neq 0$ then a monodromy map is equal to the rotation by the angle $2\pi\mu/k$. The formulas (6.6), (6.7) imply the result in the considered cases. If $\varphi k \neq 0$ and $\mu \neq 0 \bmod k$ then the monodromy map has no fixed points, hence its Lefschetz number is equal to zero. ■

Lemma 6.5.6. Let P be a hyperbolic polynomial and assume that (6.30) is satisfied. Then $(W_{P,\varphi,\varepsilon}, Y_{P,\varphi,\varepsilon}^\pm)$ is a T -periodic proper pair of ENRs and

$$\text{Lef}_T(W_{P,\varphi,\varepsilon}, Y_{P,\varphi,\varepsilon}^\pm) = \begin{cases} 1, & \text{if } \mu \neq 0 \bmod k \\ \text{Ind } P, & \text{if } \mu = 0 \bmod k \end{cases}$$

Proof: It follows by Lemmas 6.5.3 and 6.5.4 that

$(W_{P,\varphi,\varepsilon}, Y_{P,\varphi,\varepsilon}^\pm)$ is a T -periodic proper pair of ENRs. Since C_P is convex,

$$\text{Lef}_T(W_{P,\varphi,\varepsilon}) = 1.$$

The result follows by Lemma 6.5.5 and the formula (2.4). ■

Lemma 6.5.7. Let P be an elliptic polynomial and Q be a hyperbolic polynomial. Let $\delta, \psi \in \mathbb{R}$ and let l be a nonnegative integer. Assume that Q is defined by (6.11), and (6.13) holds, i.e.

$$c_{r,s} = 0 \text{ unless } r-s = 1 \bmod l.$$

Assume moreover, that (6.30) is satisfied and

$$(6.31) \quad \exists v \in \mathbb{Z}: \psi l T = 2\pi v.$$

Assume, that $\varepsilon < \frac{1}{4}$, and $\delta > 4$. Put

$$U = W_{Q,\psi,\delta} \setminus \text{int } W_{P,\varphi,\varepsilon},$$

$$V = Y_{P,\varphi,\varepsilon}^- \cup Y_{Q,\psi,\delta}^-.$$

Then (U, V) is a T -periodic proper pair of ENRs and

$$\text{Lef}_T(U, V) = \begin{cases} 0, & \text{if } \mu \neq 0 \bmod k, \nu \neq 0 \bmod l \\ 1 - \text{Ind } P, & \text{if } \mu = 0 \bmod k, \nu \neq 0 \bmod l \\ \text{Ind } Q - 1, & \text{if } \mu \neq 0 \bmod k, \nu = 0 \bmod l \\ \text{Ind } Q - \text{Ind } P, & \text{if } \mu = 0 \bmod k, \nu = 0 \bmod l \end{cases}$$

Proof. It follows by the choice of ε and δ , and by Lemma 6.5.3 that $\partial(\varepsilon C_P)$ is contained in $\{z \in \mathbb{R}^2: |z| \leq \frac{1}{2}\}$ and $\partial(\delta C_Q)$ is contained in $\{z \in \mathbb{R}^2: 2 \leq |z|\}$, hence

$$W_{P,\varphi,\varepsilon} \subseteq \text{int } W_{Q,\psi,\delta}.$$

Moreover, since P is elliptic and Q is hyperbolic, it follows also that the zero-section of U is compact. Again by Lemma 6.5.3 and by

Lemma 6.5.4, in order to prove that (U, V) is a T -periodic proper pair of ENRs it suffices to find a smooth T -periodic process such that U , $Y_{P, \varphi, \varepsilon}^{\pm}$, and $Y_{Q, \psi, \delta}^{\pm}$ consist of its trajectories. Actually, the required process is the flow ω generated by the equation

$$\dot{z} = i(a(|z|)\varphi + (1-a(|z|))\psi)z$$

where $a: [0, \infty) \longrightarrow [0, 1]$ is a smooth function such that

$a(\tau) = 1$ if $\tau \leq \frac{1}{2}$ and $a(\tau) = 0$ if $\tau \geq 2$, and is given by

$$\omega_t(z) = e^{i(a(|z|)\varphi + (1-a(|z|))\psi)t} z.$$

The zero-section of U is defined by

$$U(0) = \delta C_Q \setminus \text{int } \varepsilon C_P.$$

It is deformable to the unit circle by the radial projection.

Using that deformation it is easy to prove that the monodromy map

$\omega_T|_{U(0)}$ is homotopic to the identity, hence

$$\text{Lef}_T(U) = 0.$$

Since

$$\text{Lef}_T(U^-) = \text{Lef}_T(Y_{P, \varphi, \varepsilon}^-) + \text{Lef}_T(Y_{Q, \psi, \delta}^-),$$

by (2.4) and Lemma 6.5.5 we obtain the required formula on

$\text{Lef}_T(U, V)$. ■

6.6. Proofs of the theorems on polynomial equations. We are able to show that assumptions of Theorems 6.3.1 and 6.3.3 imply inequalities similar to (3.7), (3.8), and, as a consequence, the existence of strong periodic blocks for the considered equations.

Lemma 6.6.1. Let $m \geq 2$. Let P be an hyperbolic or elliptic polynomial given by (6.3). Assume that (6.10) is fulfilled for a

continuous, T-periodic in t function b. Assume that the conditions (6.4) and (6.30) are satisfied. Define $f(t, z)$ as the right side of (6.9). Then there exists an $\varepsilon_\infty > 0$ such that

$$(6.32) \quad \text{grad } \Lambda_{P, \varphi, \varepsilon}^-(t, z) \cdot (1, f(t, z)) > 0 \quad (\forall (t, z) \in Y_{P, \varphi, \varepsilon}^-),$$

$$(6.33) \quad \text{grad } \Lambda_{P, \varphi, \varepsilon}^+(t, z) \cdot (1, f(t, z)) < 0 \quad (\forall (t, z) \in Y_{P, \varphi, \varepsilon}^+),$$

provided $\varepsilon > \varepsilon_\infty$. Moreover, if P is hyperbolic and $\varepsilon > \varepsilon_\infty$ then $W_{P, \varphi, \varepsilon}$ is a strong T-periodic block for (6.9) and $Y_{P, \varphi, \varepsilon}^-$ is its exit set.

Proof. The existence of ε_∞ is an immediate consequence of Lemma 6.5.3, Lemma 6.5.4, and Proposition 3.4.1(i) with $H(t)z = i\varphi z$ (hence $\Omega(t)z = e^{i\varphi t}z$). If P is hyperbolic and $\varepsilon > \varepsilon_\infty$ then Lemma 6.5.6 (in fact, the argument in its proof) imply that $W_{P, \varphi, \varepsilon}$ is a strong periodic block. ■

Lemma 6.6.2. Let P and Q be hyperbolic or elliptic polynomials given by (6.3) and (6.11), respectively. Assume that $1 \leq m < n$ and the conditions (6.4), (6.13), (6.30), and (6.31) are satisfied. Let (6.14) and (6.15) hold for a continuous, T-periodic in t function b. Let $f(t, z)$ be equal to the right-hand side of (6.12). Then there exist positive numbers ε_0 and δ_∞ such that if $0 < \varepsilon < \varepsilon_0$ then (6.32), (6.33) are valid and if $\delta > \delta_\infty$ then

$$(6.34) \quad \text{grad } \Lambda_{Q, \psi, \delta}^-(t, z) \cdot (1, f(t, z)) > 0 \quad (\forall (t, z) \in Y_{Q, \psi, \delta}^-),$$

$$(6.35) \quad \text{grad } \Lambda_{Q, \psi, \delta}^+(t, z) \cdot (1, f(t, z)) < 0 \quad (\forall (t, z) \in Y_{Q, \psi, \delta}^+).$$

Moreover, if P is elliptic and Q is hyperbolic, and

$$\varepsilon < \min\{\varepsilon_0, \frac{1}{4}\}, \quad \delta > \max\{\delta_\infty, 4\}$$

then the set U defined in Lemma 6.5.7 is a strong T -periodic block for (6.12) and its exit set is equal to V .

Proof. The validity of (6.34), (6.35) is a direct consequence of Lemma 6.6.1. In order to prove (6.32), (6.33) we apply Lemma 6.5.3, Lemma 6.5.4, and Proposition 3.4.1(iii) with $H(t)z = 1\phi z$. If that P is elliptic, Q is hyperbolic, and ϵ and δ satisfy the above estimates, then U is a strong T -periodic block for (6.12) by Lemma 6.5.7 (more exactly: by the argument in the proof of Lemma 6.5.7). ■

Now it is easy to prove both the theorems on polynomial equations.

Proof of Theorem 6.3.1. The result is an immediate consequence of Lemmas 6.5.6 and 6.6.1, and Theorem 4.2.1. ■

Proof of Theorem 6.3.3. Assume that both P and Q are hyperbolic. Fix ϵ and δ as in Lemma 6.6.2, and assume in addition that $\epsilon < \frac{1}{2}$ and $\delta > 2$. By Lemma 6.5.3,

$$W_{P,\phi,\epsilon} \subseteq W_{Q,\psi,\delta},$$

hence the result follows by Lemma 6.5.6 and Theorem 4.2.2. In the case P is elliptic and Q is hyperbolic the result immediately follows by Lemmas 6.5.7 and 6.6.2, and Theorem 4.2.1. ■

6.7. Planar rational equations with periodic coefficients. We

consider the equation (1.1) in $\mathbb{R}^2 \setminus 0$ having the right-hand side of the form $\sum a_{p,q}(t)x^p\bar{x}^q$, where $p, q \in \mathbb{Z}$, and $a_{p,q}$ is T -periodic and equal to zero for almost all p and q . In that case we call (1.1) a rational equation with periodic coefficients. For any smooth function $\mathbb{R} \times (\mathbb{R}^2 \setminus 0) \longrightarrow \mathbb{R}^2$, T -periodic in the first variable, some finite expansion of the Fourier-Laurent type is of that form. In this section, based on [KS], we assume that $a_{p,q}(t) \neq 0$ for some p, q, t , $p+q < 0$.

At the beginning let us observe, that if we change the variable z into z^{-1} in the equations (6.9) and (6.12), Theorems 6.3.1 and 6.3.3 become results on periodic solutions of rational equations on $\mathbb{R}^2 \setminus 0$. Formulation of the obtained theorems is left to the reader. Our purpose is to present a result on rational equations, in which the terms z (or \bar{z}) and z^{-1} (or \bar{z}^{-1}) appear simultaneously and it cannot be reduced to the previously stated theorems. As in Chapter 6, we examine the geometry of hyperbolic and elliptic polynomials. If P is such a polynomial, the phase portrait of the local flows in $\mathbb{R}^2 \setminus 0$ generated by

$$\dot{z} = |z|^s P(z)$$

does not change for various values of $s \neq 0$. It follows that all sets and functions constructed for P in Section 6.5 preserve their properties provided $P(z)$ is replaced by $|z|^s P(z)$. In particular, the inequalities (6.21), (6.22) still hold after that replacement, hence we can use Proposition 3.4.1 in order to construct strong periodic blocks for some generated equations. Motivated by that observation we are ready to establish and prove a theorem on

rational equations.

Theorem 6.7.1. Suppose that P is elliptic polynomial of degree m given by (6.3), Q is a hyperbolic polynomial of degree n given by (6.11), and k and l are nonnegative integers such that (6.4) and (6.13) are satisfied. Let u and v be nonnegative integers, φ and ψ be real numbers, and let $b: \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) \longrightarrow \mathbb{R}^2$ be a continuous function T -periodic in the first variable. Assume that

$$m - 2u \leq 0, \quad n - 2v \geq 2,$$

$$\varphi kT = 2\pi\mu, \quad \psi lT = 2\pi\nu$$

for some $\mu, \nu \in \mathbb{Z}$, and

$$b(t, z)/|z|^{n-2v} \longrightarrow 0 \text{ as } |z| \longrightarrow \infty, \text{ uniformly in } t,$$

$$b(t, z)/|z|^{m-2u} \longrightarrow 0 \text{ as } |z| \longrightarrow 0, \text{ uniformly in } t.$$

Then the equation

$$(6.36) \quad \dot{z} = \sum_{p+q=m} a_{p,q} e^{i\varphi(q+1-p)t} z^p \bar{z}^{q-u} + \sum_{r+s=n} c_{r,s} e^{i\psi(s+1-r)t} z^r \bar{z}^{s-v} + b(t, z)$$

has a T -periodic solution provided one condition among (a), (b), and (c) in Theorem 6.3.3 is satisfied.

Proof. By the argument in the proof of Lemma 6.6.2, using Proposition 3.4.1(i), (iv) instead of (i), (iii), we conclude the existence of a strong T -periodic block for (6.36), which has the form of U in Lemma 6.5.7. By the latter lemma and by Theorem 4.2.1

the result follows. ■

Corollary 6.7.2. Assume that $k, l, p, q, r, s \in \mathbb{Z}$, $a, c, \rho, \sigma \in \mathbb{R}$, $a \neq 0$, $c \neq 0$ and for each k and l , $b_{k,l}: \mathbb{R} \longrightarrow \mathbb{C}$ is a continuous, T -periodic function. Assume moreover, that $b_{k,l}$ is equal to zero for almost all (k, l) . Then the equation

$$\dot{z} = ae^{i\rho t} \frac{1}{z^{p-q}} + \sum_{-p-q < k+l < r+s} b_{k,l}(t) z^{\frac{k-l}{2}} + ce^{i\sigma t} z^{\frac{r-s}{2}}$$

has a T -periodic solution, provided

$$p + q \geq 0, r + s \geq 2$$

and one of the following conditions is satisfied:

(a) $\rho = 0$, $\sigma = 0$, $q - p \geq 1$, $r - s \leq 1$, $q - p \neq r - s$, and T is arbitrary,

(b) $\rho \neq 0$, $\sigma = 0$, $q - p \geq 2$, and

(ba) $r - s = 1$, $T = \frac{2\pi(p+1-q)\mu}{\rho}$ for some $\mu \in \mathbb{Z}$,

(bb) $r - s \leq 0$, $T = \frac{2\pi\mu}{\rho}$ for some $\mu \in \mathbb{Z}$,

(c) $\rho = 0$, $\sigma \neq 0$, $r - s \leq 0$, and

(ca) $q - p = 1$, $T = \frac{2\pi(s+1-r)\nu}{\sigma}$ for some $\nu \in \mathbb{Z}$,

(cb) $q - p \geq 2$, $T = \frac{2\pi\nu}{\sigma}$ for some $\nu \in \mathbb{Z}$,

(d) $\rho\sigma \neq 0$, $q - p \geq 2$, $r - s \leq 0$, $T = \frac{2\pi\mu}{\rho} = \frac{2\pi\nu}{\sigma}$ for some $\mu, \nu \in \mathbb{Z}$, and

$$\mu = 0 \bmod(p+1-q) \text{ or } \nu = 0 \bmod(s+1-r).$$

Proof: Since

$$\frac{1}{z^{p-q}} = \frac{z^{q-p}}{|z|^{2q}}, \quad z^{\frac{r-s}{2}} = \frac{z^{r-s}}{|z|^{-2s}} = \frac{z^{-s-r}}{|z|^{-2r}},$$

and, by Remark 6.2.6(i), az^{q-p} is elliptic with index $q - p$ if

$q-p \geq 1$, $c\bar{z}^{-s-r}$ is hyperbolic with index $r-s$ if $s-r \geq 0$, and cz is hyperbolic with index 1, the result is a consequence of Theorem 6.7.1. ■

Example 6.7.3. The equations

$$\begin{aligned}\dot{z} &= e^{it} \left(\frac{1}{\bar{z}^2} + \bar{z}^2 \right) + 1 \\ \dot{\bar{z}} &= iz + e^{it} \left(\frac{1}{\bar{z}^2} + 1 \right)\end{aligned}$$

have 2π -periodic solutions.

Indeed, the existence of 2π -periodic solution for the first equation follows by Corollary 6.7.2(d). By the change of variables $w = \frac{1}{\bar{z}}$, the second one becomes

$$\dot{w} = -iw - e^{it} w^2 - e^{it} w^2 \bar{w},$$

hence the result follows by Corollary 6.3.5(d)(dc).

Chapter 7. Higher order equations.

In this chapter we consider scalar ordinary differential equations of order higher than one. Using the geometric method we present modifications of some results established elsewhere. In Section 7.1 we deal with second order equations, while in Section 7.2, based on the paper [SS], we consider equations of an arbitrary order.

7.1. On second order equations. We consider a scalar ordinary differential equation of second order

$$(7.1) \quad \ddot{x} = n(t, x, \dot{x}),$$

where $n: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a T -periodic (with respect to t) function of the class C^1 . (7.1) is equivalent to the planar equation

$$(7.2) \quad (\dot{x}, \dot{y}) = (y, n(t, x, y)).$$

Let R be a positive number and let $\alpha: [0, \infty) \longrightarrow (0, \infty)$ be a continuous function. Introduce another function and sets

$$\beta: [0, \infty) \ni u \longrightarrow \int_0^u \frac{s}{\alpha(s)} ds \in \mathbb{R},$$

$$C = \{(x, y) \in \mathbb{R}^2: |x| \leq R - \beta(|y|)\},$$

$$D = \{(x, y) \in C: xy \geq 0, |x| = R - \beta(|y|)\}.$$

The following proposition is a slight improvement of a classical result from the theory of second order equations presented in [RM] (compare also [GGL, IV.1] and [RSC1] for related results). Our proof is completely different from that in [RM], based on Theorem 4.1.3, and applies Theorem A.

Proposition 7.1.1 (compare [RM, Cor.5.5.4]). Assume that

$$(7.4) \quad \int_0^\infty \frac{s}{\alpha(s)} ds > R,$$

for each $t \in \mathbb{R}$,

$$n(t, -R, 0) < 0, \quad n(t, R, 0) > 0,$$

and for every $t \in \mathbb{R}$, $x \in [-R, R]$, $y \in \mathbb{R}$,

$$(7.5) \quad |n(t, x, y)| < \alpha(|y|).$$

Then $\mathbb{R} \times C$ is a T -periodic block for the local process φ generated by (7.2) and $\mathbb{R} \times D$ is its exit set. Moreover,

$$\text{ind}(\varphi_{(0, T)}, K_{\mathbb{R} \times C}) = -1.$$

In particular, there is a T -periodic solution of (7.1) contained

in $[-R, R]$. The latter assertion also holds if n is continuous and the strong inequality in (7.5) is replaced by the weak one.

Proof: By the assumption, β is a strictly increasing C^1 -function. Let $S > 0$ be such that $\beta(S) = R$ (the existence is guaranteed by (7.4)). The set $\partial C \cap \{(x, y) \in \mathbb{R}^2: x \geq 0, y \geq 0\}$ is equal to the graph of the function

$$\gamma: [0, S] \ni y \longrightarrow R - \beta(y) \in [0, R].$$

Choose a $y > 0$. The normal outward vector to the graph of γ in y is equal to

$$(1, -\gamma'(y)) = (1, \beta'(y)) = (1, y/\alpha(y)).$$

Thus, for every $\sigma \in \mathbb{R}$,

$$(1, -\gamma'(y)) \cdot (y, n(\sigma, \gamma(y), y)) \geq y(1 - |n(\sigma, \gamma(y), y)/\alpha(y)|) > 0$$

by (7.5). We have thus proved, that $\mathbb{R} \times (\partial C \cap \{x \geq 0, y > 0\})$ is contained in the exit set of $\mathbb{R} \times C$. If $\sigma \in \mathbb{R}$ then the point $(\sigma, R, 0)$ is also in the exit set. Indeed, let $\delta = (\delta_1, \delta_2)$ be the solution of (7.2) through $(\sigma, R, 0)$. Then

$$\delta'_2(\sigma) = n(\sigma, R, 0) > 0,$$

hence $\delta_2(\sigma+t) > 0$ for sufficiently small t . Thus $\delta'_1(\sigma+t) > 0$, so $\delta(\sigma+t)$ cannot intersect C for small parameters t , because both its coordinates have to increase. By repeated application of that argument for the other cases, we complete a proof of the first assertion. Obviously

$$\text{Lef}_T(\mathbb{R} \times C, \mathbb{R} \times D) = \chi(C) - \chi(D) = -1,$$

hence Theorem A establishes the main assertion. Actually $\mathbb{R} \times C$ is neither a strong nor a weak periodic block for (7.2), at least

because the condition (A3) in Definition 3.2.1 does not need to be satisfied at the points $(\pm R, 0)$, hence we cannot apply Theorem 4.2.1 to the last assertion. However, if n is continuous only and (7.5) is satisfied, it can be approximated by C^1 -functions for which (7.5) is also valid and the previous assertion apply. The resulting periodic solutions approach to the required solution of (7.1). If (7.5) is replaced by the weak inequality, we can perturb α slightly in order to obtain the strong inequality and follow the previous argument. ■

Corollary 7.1.2. Let all the assumptions of Proposition 7.1.1 be satisfied. Let

$$n(t, x, y) = ax + by + z(t, x, y),$$

and assume that for every $t \in \mathbb{R}$

$$z(t, x, y)/(x^2 + y^2)^{1/2} \longrightarrow 0 \text{ as } (x^2 + y^2)^{1/2} \longrightarrow 0 \text{ uniformly in } t.$$

If the real parts of both the eigenvalues of $\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$ are

simultaneously negative or positive, then (7.1) has a nontrivial T -periodic solution contained in $[-R, R]$.

Proof: The assumption on the matrix guarantees the asymptotical stability (positive or negative) of the origin for the linear part of (7.2). By Proposition 3.4.1(iii) we obtain a T -periodic block $\mathbb{R} \times C'$ for (10.2) with the exit set empty or, respectively, equal to $\mathbb{R} \times \partial C'$, where C' is a small ellipse containing 0 in its interior. If the trivial solution is the unique T -periodic one, Theorem B implies that in both the cases

the fixed point index of the Poincaré operator at 0 will be equal to 1, which will contradict to Proposition 7.1.1. ■

Example 7.1.3. Consider a Duffing-type equation

$$\ddot{x} = \dot{x} - x + x^3 + \lambda \sin(x^2 + (\dot{x})^2) p(t).$$

Assume that p is a continuous T -periodic function. Then for every $\lambda \neq 0$ the equation has a nonzero T -periodic solution.

Indeed, by the argument in [RM, p.195] we conclude that the assumptions of Proposition 7.1.1 are satisfied, hence Corollary 7.1.2 implies the result.

7.2. Results on n -th order equations. The problem of the existence of periodic solutions of nonlinear nonautonomous n -th order equations has been studied in various papers. Earlier results are presented in book [RSC2], which contains an extensive bibliography on the problem. A bibliography of more recent results is presented in [OZ].

The equation considered here has the form:

$$(7.6) \quad y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' = b(t, y, y', \dots, y^{(n-1)})$$

where a_1, \dots, a_{n-1} denote real numbers and $b: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is a continuous map. Let T be a positive number and let ε denote the number 1 or -1. In the sequel we will impose the following conditions on a_1, \dots, a_{n-1}, b and T :

(D1) $b(\cdot, x_1, \dots, x_n)$ is T -periodic for each $(x_1, \dots, x_n) \in \mathbb{R}^n$,

(D2) The equation

$$(7.7) \quad \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \dots + a_2\lambda + a_1 = 0$$

has no pure imaginary roots,

$$(D3) \quad \frac{b(t, x_1, \dots, x_n)}{|(x_2, \dots, x_n)|} \longrightarrow 0 \text{ as } |(x_2, \dots, x_n)| \longrightarrow \infty \text{ uniformly}$$

in $(t, x_1) \in \mathbb{R}^2$,

$$(D4) \quad \varepsilon b(t, x_1, \dots, x_n) x_1 \longrightarrow \infty \text{ as } |x_1| \longrightarrow \infty \text{ uniformly in } (t, x_2, \dots, x_n) \text{ on compact subsets of } \mathbb{R}^n.$$

Our first result is not original and is a consequence of a more general theorem due to J.R.Ward. We present it here, because our proof (based on Theorem 4.2.1) differs from the original one (which apply Theorem 4.1.3) and can be given simultaneously with the proof of the second result.

Proposition 7.2.1 (compare [W1, Th.1]) The equation (7.6) has a T-periodic solution provided the conditions (D1) - (D4) hold.

Proposition 7.2.2. Assume (D1) - (D4). Assume in addition that

$$b(t, x_1, \dots, x_n) = b_1 x_1 + \dots + b_n x_n + c(t, x_1, \dots, x_n)$$

with c continuous satisfying the condition

$$\frac{c(t, x_1, \dots, x_n)}{|(x_1, \dots, x_n)|} \longrightarrow 0 \text{ as } |(x_1, \dots, x_n)| \longrightarrow 0 \text{ uniformly in } t \in \mathbb{R},$$

and constants b_1, \dots, b_n such that the equation

$$(7.8) \quad \lambda^n + (a_{n-1} - b_n) \lambda^{n-1} + \dots + (a_1 - b_2) \lambda - b_1 = 0$$

has no roots on $i\mathbb{R}$. Denote by k and l the number of roots (counted with their multiplicities) of, respectively, (7.7) and (7.8) which have real positive parts. If

$$(-1)^k (\text{sgn } a_1) \varepsilon = (-1)^l$$

then (7.6) has a nontrivial T-periodic solution. If, moreover, b is odd in x, i.e.

$$b(t, -x_1, \dots, -x_n) = -b(t, x_1, \dots, x_n)$$

then there are at least two distinct such solutions.

Proof of Propositions 7.2.1 and 7.2.2. Rewrite the equation (7.6) in an equivalent form:

$$(7.9) \quad x' = Ax + B(t, x)$$

where $x = \text{col}(x_1, \dots, x_n)$, A is the nxn-matrix defined by

$$A = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}$$

and $B(t, x) = \text{col}(0, \dots, 0, b(t, x))$. Consider an auxiliary equation

$$(7.10) \quad u' = Pu$$

where P is an (n-1)x(n-1)-matrix,

$$P = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}$$

By (D2),

$$(7.11) \quad \mathbb{R}^{n-1} = U_+ \oplus U_-$$

where U_+ (and U_-) is the generalized eigenspace of the eigenvalues with positive real parts (negative real parts, respectively). In order to simplify notation, suppose that both U_+ and U_- are different from $\{0\}$. (The case U_+ or U_- equal to \mathbb{R}^n follows by the similar argument; it has been also considered in [S1,2].) Let $\dim U_+ = k$ and let

$$P_+ = P|_{U_+}, \quad P_- = P|_{U_-},$$

hence

$$P = \text{diag}(P_+, P_-).$$

For an element $u \in \mathbb{R}^{n-1}$ denote by u_+ and u_- its components in the decomposition (7.11). For a $\lambda > 0$ define

$$\psi_+^\lambda(u) = \phi_+(\lambda^{-1}u_+) - 1,$$

$$\psi_-^\lambda(u) = \phi_-(\lambda^{-1}u_-) - 1.$$

where ϕ_+ , ϕ_- denote quadratic forms being Lyapunov functions associated with the equations

$$v' = -P_+v$$

$$w' = P_-w$$

respectively. Since the right-hand side of (7.10) is homogeneous of degree 1, by (D3) and Proposition 3.4.1(ii), there exists a $\mu > 0$ such that for any $(t, y) \in \mathbb{R}^2$

$$(7.12) \quad \text{grad } \psi_+^\mu(u) \cdot (P(u) + \text{col}(0, \dots, 0, b(t, y, u))) > 0,$$

$$(\forall u: \psi_+^\mu(u) = 0, \psi_-^\mu(u) \leq 0),$$

$$(7.13) \quad \text{grad } \psi_-^\mu(u) \cdot (P(u) + \text{col}(0, \dots, 0, b(t, y, u))) < 0,$$

$$(\forall u: \psi_+^\mu(u) \leq 0, \psi_-^\mu(u) = 0).$$

Set

$$C = \{u \in \mathbb{R}^{n-1}: \psi_+^\mu(u) \leq 0, \psi_-^\mu(u) \leq 0\}.$$

Define a function $\omega: \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\omega(x_1, \dots, x_n) = ((a_2x_2 + \dots + a_{n-1}x_{n-1} + x_n)x_1 + \frac{a_1}{2}x_1^2)$$

The direct calculation yields

$$(7.14) \quad \text{grad } \omega(x) \cdot (Ax + B(t, x)) =$$

$$a_2x_2^2 + a_3x_3x_2 + \dots + a_{n-1}x_{n-1}x_2 + b(t, x)x_1.$$

Since C is compact, there exists an $M < \infty$ such that

$$M \geq \max \{|a_2 x_2^2 + a_3 x_3 x_2 + \dots + a_{n-1} x_{n-1} x_2| : (x_2, \dots, x_n) \in C\}.$$

By (D1) and (D4), there exists an $R > 0$ such that if $|x_1| \geq R$ then

$$(7.15) \quad \varepsilon b(t, x) x_1 > M \quad (\forall (t, x_2, \dots, x_n) \in \mathbb{R} \times C).$$

Denote

$$N = \max \{|a_2 x_2 + \dots + a_{n-1} x_{n-1} + x_n| : (x_2, \dots, x_n) \in C\}$$

Let $r > 0$ be so chosen that

$$(7.16) \quad \frac{\sqrt{2|a_1|r} - N}{|a_1|} > R$$

and define as in [S1,2] a function $L_2: \mathbb{R}^n \longrightarrow \mathbb{R}$ by the formula

$$L_2(x) = (\operatorname{sgn} a_1) \omega(x) - r$$

From the formula for the roots of the equation $L_2(x) = 0$

$$(7.17) \quad x_1^\pm = \frac{\pm \sqrt{\Delta} - (\operatorname{sgn} a_1)(a_2 x_2 + \dots + a_{n-1} x_{n-1} + x_n)}{|a_1|},$$

where

$$\Delta = (a_2 x_2 + \dots + a_{n-1} x_{n-1} + x_n)^2 - 2|a_1|r$$

and (7.16) it is clear that for $(x_2, \dots, x_n) \in C$, the roots x_1^\pm are different from zero and of opposite signs. Moreover, $|x_1^\pm| > R$.

Define also functions $L_1, L_3: \mathbb{R}^n \longrightarrow \mathbb{R}$:

$$L_1(x) = \psi_+^\mu(x_2, \dots, x_n),$$

$$L_3(x) = \psi_-^\mu(x_2, \dots, x_n),$$

and let E be the set

$$E = \{x \in \mathbb{R}^n : L_i(x) \leq 0, i = 1, 2, 3\}.$$

Its boundary Γ is the union of sets

$$\Gamma_i = \{x \in E : L_i(x) = 0\},$$

$i = 1, 2, 3$. By (7.12) and (7.13),

$$(7.18) \quad \operatorname{grad} L_1(x) \cdot (Ax + B(t, x)) > 0 \quad (\forall x \in \Gamma_1),$$

$$(7.19) \quad \text{grad } L_3(x) \cdot (Ax + B(t, x)) < 0 \quad (\forall x \in \Gamma_3),$$

and by (7.14), (7.15), and (7.18)

$$(7.20) \quad (\text{sgn } a_1)\varepsilon \text{ grad } L_2(x) \cdot (Ax + B(t, x)) > 0 \quad (\forall x \in \Gamma_2).$$

The set $\mathbb{R} \times E$ is thus a strong T -periodic block for (7.9). Assume that $(\text{sgn } a_1)\varepsilon = 1$. The exit set of $\mathbb{R} \times E$ is given by

$$(\mathbb{R} \times E)^- = \mathbb{R} \times (\Gamma_1 \cup \Gamma_2).$$

The set $\Gamma_1 \cup \Gamma_2$ is homeomorphic to $\{0, 1\} \times C \cup [0, 1] \times S^{k-1} \times B^{n-k-1}$, hence it has the homotopy type of S^k . By (7.17), E is homeomorphic to $[0, 1] \times C$, and, in consequence, it is also homeomorphic to the n -dimensional ball B^n . It follows that

$$\text{Lef}_T(\mathbb{R} \times E, (\mathbb{R} \times E)^-) = \chi(E) - \chi(\Gamma_1 \cup \Gamma_2) = (-1)^{k+1}.$$

If $(\text{sgn } a_1)\varepsilon = -1$ then

$$(\mathbb{R} \times E)^- = \mathbb{R} \times \Gamma_1.$$

hence it has the homotopy type of S^{k-1} . We conclude, that in both the cases

$$(7.21) \quad \text{Lef}_T(\mathbb{R} \times E, (\mathbb{R} \times E)^-) = (-1)^{k+1}(\text{sgn } a_1)\varepsilon.$$

By Theorem 4.2.1, Proposition 7.2.1 is proved.

We continue the proof under assumptions of Theorem 7.2.2. By a similar argument as above, using Lyapunov functions and Proposition 3.4.1(iii) one can construct functions

$M_1, M_2: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the set

$$E_0 = \{x \in \mathbb{R}^n: M_1(x) \leq 0, M_2(x) \leq 0\}$$

is a compact neighborhood of 0 homeomorphic to B^n and contained in E , satisfying inequalities

$$\text{grad } M_1(x) \cdot (Ax + B(t, x)) > 0, \quad (\forall x \in E_0, M_1(x) = 0),$$

$$\text{grad } M_2(x) \cdot (Ax + B(t, x)) < 0, \quad (\forall x \in E_0, M_2(x) = 0),$$

and $\mathbb{R} \times E_0$ is a strong T -periodic block for (7.9) with the exit set given by

$$(\mathbb{R} \times E_0)^- = \mathbb{R} \times \{x \in E_0 : M_1(x) = 0\}.$$

Since it the homotopy type of S^{1-1} , we conclude that

$$\text{Lef}_T(\mathbb{R} \times E_0, (\mathbb{R} \times E_0)^-) = (-1)^1.$$

The first assertion of Proposition 7.2.2 follows by Theorem 4.2.2.

The last conclusion concerning the existence of two distinct periodic solutions follows easily from the fact, that since b is odd with respect to x , the set of solutions of (7.6) is invariant under multiplication by -1 . The proof is finished. ■

Example 7.2.3. The equation

$$y^{(3)} - y' = 2y e^{-y^2} - \arctan y + \sin(y^3) \sin t$$

has two distinct nonzero 2π -periodic solutions.

Indeed, in that case $a_1 = \varepsilon = -1$ and $k = 1 = 1$.

FINAL COMMENTS

All results on periodic trajectories of T -periodic processes on a space X can be translated into results on periodic trajectories of flows on $S^1 \times X$ via the identification $S^1 = \mathbb{R}/T\mathbb{Z}$. After that identification T -periodic blocks become usual (i.e. compact) blocks in the new phase space, they have the form of total spaces of locally trivial bundles over S^1 . There is also a reverse translation, provided the phase space of a flow has an angle coordinate and the flow moves along that coordinate with a positive speed. Actually, that assumption on the flow seems to be necessary for results on the existence of a periodic orbit in a given set, unless the phase space is a 2-dimensional manifold (hence the Poincaré-Bendixson theory is valid). As it was shown in [Sc], even a flow in a 3-dimensional solid torus does not need to have a periodic orbit. Theorems A, B, and C can be applied to flows with a positive speed in an angle coordinate, however we have not investigated those applications here.

In Chapter 6 we have established several results on periodic solutions of some concrete polynomial and rational equations. A proper choice of coefficients in those equations was essential for a possibility of application of the geometric method. It is of interest to know whether periodic solutions exist for some other choice of coefficients. In particular, we assumed that the coefficient in the higher order term is of the form $ae^{i\varphi t}$ for some real numbers a and φ , hence we are interested in replacing that

form by a finite sum $\sum a_k e^{i\varphi t/k}$, like in general Fourier-Taylor polynomials. Another problem is to find a relation between the coefficients and the elements of braid groups determined by periodic solutions. If a periodic solution is contained in $\mathbb{R}^2 \setminus 0$, there arises a problem of determination of the homotopy class of its loop. In particular, if that class is nontrivial, an isolated periodic solution of a rational equation with a singularity at the origin cannot be continued to a zero of any planar vector-field, hence Theorem 4.1.4 does not apply to it. That problem is also connected with determination of the Nielsen number of the Poincaré operator inside a periodic block.

Results presented in this paper concern ordinary differential equations only. It is a natural problem to extend them to other classes of equations generating processes, like functional differential equations, neutral equations or semilinear parabolic partial differential equations. Extensions of the results to equations equivariant with respect to a compact Lie group action or extensions to differential inclusions (or, more generally, to multivalued processes) can also be considered. Another question is to find counterparts of the geometric method for boundary value problems other than the periodic one. The method can be applied to results on bifurcations of periodic solutions. We left all those topics to further research.

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